Group analysis of an ideal plasticity model

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Abstract. In this paper, we study the Lie point symmetry group of a system describing an ideal plastic plane flow in two dimensions in order to find analytical solutions of the system. The infinitesimal generators that span the Lie algebra for this system are obtained, six of which are original to this paper. We completely classify the subalgebras of codimension one and two into conjugacy classes under the action of the symmetry group. We apply the symmetry reduction method in order to obtain invariant and partially invariant solutions. These solutions are of algebraic, trigonometric, inverse trigonometric and elliptic type. Some solutions, depending on one or two arbitrary functions of one variable, have also been found. In some cases, the shape of a potentially feasible extrusion die corresponding to the solution is deduced. These tools could be used to curve and undulate rectangular rods or slabs, or to shape a ring in an ideal plastic material.

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1. Introduction

In this paper, we investigate the plane flow of ideal plastic materials [2, 3, 4] modelled by the hyperbolic system of four partial differential equations (PDE) in $q = 4$ dependent variables σ, θ, u, v and $p = 2$ independent variables x and y,

$$
\begin{aligned}
&(a) \qquad && \sigma_x - 2k\,(\theta_x \cos 2\theta + \theta_y \sin 2\theta) = 0,\\
&(b) \qquad && \sigma_y - 2k\,(\theta_x \sin 2\theta - \theta_y \cos 2\theta) = 0,\\
&(c) \qquad && (u_y + v_x)\sin 2\theta + (u_x - v_y)\cos 2\theta = 0,\\
&(d) \qquad && u_x + v_y = 0,
\end{aligned}
\tag{1}
$$

where $\sigma_x = \partial\sigma/\partial x$, *etc.* The expressions (1.a), (1.b), are the equilibrium equations for the plane problem. In other words they are the Cauchy differential equations of motion in a continuous medium where we consider that the sought quantities do not depend on z. These two equations involve the dependent variables σ and θ that define the stress tensor; σ is the mean pressure and θ is the angle relative to the x axis in the counterclockwise direction minus $\pi/4$. The equation (1.c) corresponds, in the plane case, to the Saint-Venant-Von Mises plasticity theory equations, where u and v are respectively the velocities in the x axis and y axis directions. Moreover, we assume that the material is incompressible and hence that the velocity vector is divergenceless. This explains the presence of the equation (1.d) in the considered system. The positive-definite constant k is called the yield limit and it is associated with the plastic material. Without loss of generality we assume that $k = 1/2$ (this is the same as re-scaling the pressure σ).

In a recent work [18], the concept of homotopy of two functions has been used to construct two families of exact solutions for the system formed by the two first equations in (1). In the papers [5, 6], the Nadaï solution [7] for a circular cavity under normal stress and shear and the Prandtl solution [8] for a bloc compressed between two plates has been mapped by elements of a symmetry group of the system consisting of (1.a) and (1.b), in order to calculate new solutions. In addition in [9], simple and double Riemann wave solutions for the system (1) were found using the method of characteristics. However, as is often the case with this method, those solutions rely on numerical integration for obtaining the velocities u and v. Symmetries of the system (1) were found in [1]. However, the Lie algebra of symmetries, was not complete because of the absence of the generators B_1, B_3, B_4, B_4, B_6 and K (or equivalents) defined by (4) in Section 2 of this work. Moreover, we found two infinite-dimensional subalgebras spanned by X_1 and X_2 provided below in equation (5). The generator X_1 was known [1] as a symmetry of the two first equations in (1), but it is shown in this paper to also be a symmetry of the complete system (1). The generator X_2 is a new one. A classification of one-dimensional subalgebras was performed in [1]. To our knowledge, no systematic Lie group analysis based on a complete subalgebra classification in conjugacy classes under the action of

the symmetry group G of the system (1) that includes the new found generators has been done before.

The goal of this paper is to systematically investigate the system (1) from the perspective of the Lie group of point symmetries G in order to obtain analytical solutions. That is, we obtain in a systematic way all invariant and partially invariant (of structure defect $\delta = 1$ in the sense defined by Ovsiannikov [10]) solutions under the action of G which are non-equivalent. Invariant solutions are said to be non-equivalent if they cannot be obtained from one another by a transformation of G (the solutions are not in the same orbit). In practice, we apply a procedure developed by J. Patera *et al.* [11, 12, 13] that consists of classifying the subalgebras of $\mathcal{L}$ associated with G into conjugacy classes under the action of G. Two subalgebras $\mathcal{L}_i \subset \mathcal{L}$ and $\mathcal{L}'_i \subset \mathcal{L}$ are conjugate if $G\mathcal{L}_iG = \mathcal{L}'_i$. For each conjugacy class, we choose a representative subalgebra, find its invariants and use them to reduce the initial system (1) to a system in terms of the invariants which involve fewer variables. We illustrate these theoretical considerations with some classes of algebraic solutions, some of them in closed form and others defined implicitly. Thereafter, we draw for some solutions and specific choice of parameters the shape of the corresponding extrusion die. The applied method relies on the fact that the contours of the tools must coincide with the flow lines described by the velocities u and v of the solutions of the problem. For applications, it is convenient to feed material into the extrusion die rectilinearly at constant speed. So, the tools illustrated in this paper were drawn considering this kind of feeding. Based on mass conservation and on the incompressibility of the materials, we easily deduce that the curve defining the limit of the plasticity region for constant feeding speed must obey the ordinary differential equation (ODE)

$$\frac{dy}{dx} = \frac{V_0 - v(x,y)}{U_0 - u(x,y)}, \tag{2}$$

where U_0, V_0 are components of the feeding velocity of the die (or extraction velocity at the output of the die) respectively along the x-axis and y-axis. One should note that the conditions (2) are reduced to those required on the limits of the plasticity region in reference [9] when $V_0 = 0$ and that the curves defining the limits coincide with slip lines (characteristics), that is when we require

$$dy/dx = \tan\theta(x,y) \text{ or } dy/dx = -\cot\theta(x,y). \tag{3}$$

Thus the condition (2) can be viewed as a relaxation of the boundary conditions given in [9]. The reason we can use these relaxed conditions is that we choose the contours of the tool to coincide with the flow lines for a given solution rather than require the flow of material to be parallel to the contours. Using these relaxed conditions, we can choose (in some limits) the feeding speed and direction for a tool and this determines the limits of the plasticity region.

This paper is organized as follows. In section 2 we give the infinitesimal generators spanning the Lie algebra of symmetries $\mathcal{L}$ for the system (1) and the discrete transformations leaving it invariant. A brief discussion on the classification of subalgebras of $\mathcal{L}$ in conjugacy classes follows. Section 3 is concerned with symmetry reduction. It describes how the symmetry reduction method (SRM) has been applied to the system (1) and the method for finding partially invariant solutions (PIS). Some examples of solutions are presented, including invariant and partially invariant ones. We conclude this paper with a discussion on the obtained results and some future outlook.

2. Symmetry algebra and classification of its subalgebras

In this section we study the symmetries of the system (1). Following the standard algorithm [15], the Lie symmetry algebra $\mathfrak{L}$ of the system has been determined. Using the notation $\partial_x = \partial/\partial_x$, *etc.*, the Lie algebra of symmetry is spanned by the fourteen infinitesimal generators

$$\begin{gathered}
P_1 = \partial_x, \quad P_2 = \partial_y, \quad P_3 = \partial_u, \quad P_4 = \partial_v, \quad P_5 = \partial_\sigma, \\
D_1 = x\partial_x + y\partial_y + u\partial_u + v\partial_v, \quad D_2 = x\partial_x + y\partial_y - u\partial_u - v\partial_v, \\
L = -y\partial_x + x\partial_y - v\partial_u + u\partial_v + \partial_\theta \\
B_1 = -v\partial_x + u\partial_y, \quad B_2 = y\partial_u - x\partial_v, \\
B_3 = (\sigma + \tfrac{1}{2}\sin 2\theta)\partial_x - \tfrac{1}{2}\cos 2\theta\ \partial_y, \qquad B_4 = -\tfrac{1}{2}\cos 2\theta\ \partial_x + (\sigma - (1/2)\sin 2\theta)\partial_y, \\
B_5 = (\sigma - \tfrac{1}{2}\sin 2\theta)\partial_u + \tfrac{1}{2}\cos 2\theta\ \partial_v, \qquad B_6 = \tfrac{1}{2}\cos 2\theta\ \partial_u + (\sigma + \tfrac{1}{2}\sin 2\theta)\partial_v, \\
\begin{aligned}
K =&(\tfrac{1}{2}x\cos 2\theta - y(\sigma + \tfrac{1}{2}\sin 2\theta))\partial_x + ((\sigma - \tfrac{1}{2}\sin 2\theta)x + \tfrac{1}{2}y\cos 2\theta)\partial_y \\
&+ (\tfrac{1}{2}u\cos 2\theta + v(\tfrac{1}{2}\sin 2\theta - \sigma))\partial_u + ((\sigma + \tfrac{1}{2}\sin 2\theta)u - \tfrac{1}{2}v\cos 2\theta)\partial_v \\
&+ \theta\partial_\sigma + \sigma\partial_\theta,
\end{aligned}
\end{gathered} \tag{4}$$

and the two generators

$$X_1 = \xi(\sigma,\theta)\partial_x + \eta(\sigma,\theta)\partial_y, \qquad X_2 = \phi(\sigma,\theta)\partial_u + \psi(\sigma,\theta)\partial_v, \tag{5}$$

where the coefficients ξ and η must satisfy the two quasilinear PDEs of the first order

$$\xi_\sigma = \cos 2\theta\xi_\theta + \sin 2\theta\eta_\theta, \qquad \xi_\theta = \cos 2\theta\xi_\sigma + \sin 2\theta\eta_\sigma, \tag{6}$$

while the coefficients ϕ and ψ must solve the two PDEs, of the same type as the previous ones,

$$\phi_\sigma = -\left(\cos 2\theta\phi_\theta + \sin 2\theta\psi_\theta\right), \qquad \phi_\theta = -\left(\cos 2\theta\phi_\sigma + \sin 2\theta\psi_\sigma\right). \tag{7}$$

Note that X_1 and X_2 span infinite subalgebras. The generators D_1 and D_2 generate dilations in the space of the independent variables $\{x, y\}$ and dependent variables $\{u, v\}$. The generator L corresponds to a rotation. Moreover, B_i, $i = 1, \ldots, 4$ are associated with a type of boost and the P_i, $i = 1, \ldots, 5$ generate translations. Of these fourteen generators eight were already known [1], but six are original to this paper. The newly

found generators are $B_1, B_3 - B_6$ and K. In addition, the system (1) admits two infinite dimensional subalgebras. The one spanned by X_1 was known [1] as a symmetry of the two first equations of the system (1), but it still a symmetry of the entire system (1). The infinite subalgebra corresponding to X_2 is original to this paper. The commutation relations for the 14−dimensional Lie subalgebra

$$\mathcal{L} = \{B_1, D_2, B_2, L, D_1, B_3, B_4, B_5, B_6, P_1, P_2, P_3, P_4, P_5\}\,, \tag{8}$$

excluding K, are shown in table 1. Note that the generators P_1, P_2, B_3, B_4 are just

Table 1. Commutation relations for the algebra $\mathcal{L}$ excluding K in equation (4).

$\mathcal{L}$	B_1	D_2	B_2	D_1	L	P_5	B_3	B_4	B_5	B_6	P_1	P_2	P_3	P_4
B_1	0	$2B_1$	$-D_2$	0	0	0	0	0	$-B_4$	B_3	0	0	$-P_2$	P_1
D_2	$-2B_1$	0	$2B_2$	0	0	0	$-B_3$	$-B_4$	B_5	B_6	$-P_1$	$-P_y$	P_3	P_4
B_2	D_2	$-2B_2$	0	0	0	0	B_6	$-B_5$	0	0	P_4	$-P_3$	0	0
D_1	0	0	0	0	0	0	$-B_3$	$-B_4$	$-B_5$	$-B_6$	$-P_1$	$-P_2$	$-P_3$	$-P_4$
L	0	0	0	0	0	0	$-B_4$	B_3	$-B_6$	B_5	$-P_2$	P_1	$-P_4$	P_3
P_5	0	0	0	0	0	0	P_1	P_2	P_3	P_4	0	0	0	0
B_3	0	B_3	$-B_6$	B_3	B_4	$-P_1$	0	0	0	0	0	0	0	0
B_4	0	B_4	B_5	B_4	$-B_3$	$-P_2$	0	0	0	0	0	0	0	0
B_5	B_4	$-B_5$	0	B_5	B_6	$-P_3$	0	0	0	0	0	0	0	0
B_6	$-B_3$	$-B_6$	0	B_6	$-B_5$	$-P_4$	0	0	0	0	0	0	0	0
P_1	0	P_1	$-P_4$	P_1	P_2	0	0	0	0	0	0	0	0	0
P_2	0	P_2	P_3	P_2	$-P_1$	0	0	0	0	0	0	0	0	0
P_3	P_2	$-P_3$	0	P_3	P_4	0	0	0	0	0	0	0	0	0
P_4	$-P_1$	$-P_4$	0	P_4	$-P_3$	0	0	0	0	0	0	0	0	0

particular cases of X_1 in (6) while P_3, P_4, B_5, B_6 are particular cases of X_2. Nevertheless, we include them in the classification under conjugacy classes that we consider in the sequel. We exclude K from $\mathcal{L}$ because it cannot span a finite Lie algebra together with $P_1, P_2, P_3, P_4, B_3, B_4, B_5, B_6$. The maximal finite symmetry Lie algebra that includes K is

$$\mathcal{S} = \{B_1, D_2, B_2, K, L, P_5, D_1\}\,. \tag{9}$$

The commutation relations for $\mathcal{S}$ are given in table 2. One should note that the system

Table 2. Commutation relations for the algebra $\mathcal{S}$.

$\mathcal{S}$	B_1	D_2	B_2	K	L	P_5	D_1
B_1	0	$2B_1$	$-D_2$	0	0	0	0
D_2	$-2B_1$	0	$2B_2$	0	0	0	0
B_2	D_2	$-2B_2$	0	0	0	0	0
K	0	0	0	0	$-P_5$	$-L$	0
L	0	0	0	P_5	0	0	0
P_5	0	0	0	L	0	0	0
D_1	0	0	0	0	0	0	0

(1) is also invariant under the discrete transformations:

$$\begin{aligned} R_1 :\ & x \mapsto -x, && y \mapsto -y, && \sigma \mapsto \sigma, && \theta \mapsto \theta, && u \mapsto u, && v \mapsto v; \\ R_2 :\ & x \mapsto x, && y \mapsto y, && \sigma \mapsto \sigma, && \theta \mapsto \theta, && u \mapsto -u, && v \mapsto -v. \end{aligned} \tag{10}$$

These transformations R_1 and R_2 are rotations by an angle π in the plane of independent variables x, y and of dependent variables u, v respectively that induce the automorphisms of the Lie algebra $\mathcal{L}$:

$$\begin{aligned}
\mathcal{R}_1 : & D_1 \mapsto D_1, \quad D_2 \mapsto D_2, \quad B_i \mapsto -B_i,\ i = 1..4, \quad B_i \mapsto B_i,\ i = 5, 6, \\
& P_i \mapsto -P_i,\ i = 1, 2, \quad P_i \mapsto P_i,\ i = 3, 4, 5, \quad L \mapsto L, \quad K \mapsto K; \\
\mathcal{R}_2 : & D_1 \mapsto D_1, \quad D_2 \mapsto D_2, \quad B_i \mapsto -B_i,\ i = 1, 2, 5, 6, \quad B_i \mapsto B_i,\ i = 3, 4, \\
& P_i \mapsto P_i,\ i = 1, 2, 5 \quad P_i \mapsto -P_i,\ i = 3, 4, \quad L \mapsto L, \quad K \mapsto K.
\end{aligned} \tag{11}$$

Since we look for solutions that are invariant and partially invariant of defect structure $\delta = 1$, we only have to classify the subalgebras of codimension 1 and 2. We consider separately the two distinct finite dimensional symmetry Lie algebras discussed above.

2.1. Classification into conjugacy classes of the subalgebra $\mathcal{L}$.

In order to apply the method of classification into conjugacy classes developed in [11, 12], one first chooses a decomposition of the Lie algebra. We now describe the decomposition used for the classification of algebra $\mathcal{L}$. We begin by factoring $\mathcal{L}$ into the semi-direct sum of the one-dimensional subalgebra $\{P_5\}$ and the ideal

$$\mathcal{M} = \{B_1, D_2, B_2, L, D_1, B_3, B_4, B_5, B_6, P_1, P_2, P_3, P_4\},$$

i.e.

$$\mathcal{L} = \{P_5\} \triangleright \mathcal{M}. \tag{12}$$

Before we can apply the classification procedure to the semi-direct sum (12), we have to know the classification of the subalgebras $\{P_5\}$ and M. The subalgebra $\{P_5\}$ is a representative of its conjugacy class. In order to classify M, consider the following decomposition:

$$\mathcal{M} = \mathcal{F} \triangleright \mathcal{N}, \tag{13}$$

where $\mathcal{F} = \{B_1, D_2, B_2, L, D_1\}$ and $\mathcal{N} = \{B_3, B_4, B_5, B_6, P_1, P_2, P_3, P_4\}$ is an Abelian subalgebra. The subalgebra F is further decomposed into the direct sum

$$\mathcal{F} = \mathcal{A} \oplus \mathcal{R}, \tag{14}$$

of a simple algebra $\mathcal{A} = \{B_1, D_2, B_2\}$ and an Abelian algebra $\mathcal{R} = \{L, D_1\}$. The classification of the simple algebra was carried out by Patera and Winternitz in their work [16] on the classification of subalgebras of real Lie algebras in three and four dimensions. For the classification of subalgebra R, the conjugacy classes are simply its subspaces since R is Abelian. We then use the Goursat twist to obtain a list of representative subalgebras for the conjugacy classes of F. The subalgebra N is Abelian, so, once again, all subspaces are representative subalgebras. Therefore, any further decomposition may seem superflous. However, since we factored out the infinitesimal generator P_5 and it does not appear in M, subalgebras of type $\mathcal{B} \oplus \{0\}$ and $\{0\} \oplus \mathcal{P}$,

where $\mathcal{B} = \{B_3, B_4, B_5, B_6\}$ and $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$, have the same algebraic structure in M, see table 1. That is, they have the same commutation relations with elements of F. Therefore, taking the decomposition

$$\mathcal{N} = \mathcal{B} \oplus \mathcal{P},$$

a classification of the splitting subalgebras of the form $\mathcal{F}_i \triangleright \{\mathcal{B} \oplus \{0\}\}$, where $\mathcal{F}_i \subset \mathcal{F}$ is a representative subalgebra of the classification of $\mathcal{F}$, will lead to that of the splitting subalgebras of form $\mathcal{F}_i \triangleright \{\{0\} \oplus \mathcal{P}\}$ when (B_3, B_4, B_5, B_6) is relabelled (P_1, P_2, P_3, P_4). Since a splitting subalgebra is a subspace of form $\mathcal{F}_i \times \mathcal{B}_i$ in which $\mathcal{B}_i \subset \mathcal{B}$ is an ideal, the splitting subalgebras with non-zero component in $\mathcal{B}$ and in $\mathcal{P}$ are classified as follows. For each splitting subalgebra of form $\mathcal{F}_i \triangleright \{\mathcal{B}_j \oplus \{0\}\}$ whose basis vectors $\mathcal{B}_j$ are labeled b_s, $s = 1, \ldots, n = \dim \mathcal{B}_j$, one adds to each basis vector b_s an element p_s of $\mathcal{P}$ under the most general form $p_s = \sum_{t=1}^{4} \mu_{st} P_t$, $\mu_{st} \in \mathbb{R}$. Next, we require that the space generated by $\{b_s + p_s\}$, $s = 1, \ldots, n$ be an ideal in $\mathcal{F}_i \times \{b_s + p_s\}$. This leads to constraints on the parameters μ_{st}. Once these conditions have been satisfied, one reduces the range of these parameters as much as possible by conjugation with $Nor(F_i \triangleright \{\mathcal{B}_j \oplus \{0\}\}\,; \exp \mathcal{M})$. This completes the classification of the splitting subalgebras of M. From the list of splitting subalgebras, we use the procedure described in [17] to find the subalgebras which are not conjugate to any splitting subalgebras, the so-called nonsplitting subalgebras. Finally, we have to classify the semi-direct sum $\{P_5\} \triangleright \mathcal{M}$, which we do by using the semi-direct sum as developed in [11, 12]. Next, we reduce the range of the parameters as much as possible using the algebra automorphisms (11). The result of the classification of the one-dimensional subalgebras is presented in table 3. The classification of conjugacy classes for two-dimensional subalgebras consists of a list of 250 representative subalgebras. This list can be found in the Appendix.

$\mathcal{L}_{1,1} = \{B_1\}$	$\mathcal{L}_{1,19} = \{D_2 - D_1 + B_3 + aP_2 + \delta P_5\}$	$\mathcal{L}_{1,37} = \{B_3 + B_5 + \epsilon P_1 + aP_2\}$
$\mathcal{L}_{1,2} = \{B_1 + L + aD_1\}$	$\mathcal{L}_{1,20} = \{D_2 - D_1 + P_1 + aP_5\}$	$\mathcal{L}_{1,38} = \{B_3 + B_5 + \epsilon P_2\}$
$\mathcal{L}_{1,3} = \{B_1 + L + aD_1 + \delta P_5\}$	$\mathcal{L}_{1,21} = \{D_2 + D_1 + B_5\}$	$\mathcal{L}_{1,39} = \{B_3 + B_5 + \epsilon P_5\}$
$\mathcal{L}_{1,4} = \{B_1 + D_1\}$	$\mathcal{L}_{1,22} = \{D_2 + D_1 + P_3 + aP_5\}$	$\mathcal{L}_{1,40} = \{B_3 + \delta P_2\}$
$\mathcal{L}_{1,5} = \{B_1 + D_1 + \delta P_5\}$	$\mathcal{L}_{1,23} = \{D_2 + D_1 + B_5 + \delta P_4\}$	$\mathcal{L}_{1,41} = \{B_3 + \delta P_2 + \epsilon P_4\}$
$\mathcal{L}_{1,6} = \{B_1 + B_5\}$	$\mathcal{L}_{1,24} = \{D_2 + D_1 + B_5 + aP_4 + \delta P_5\}$	$\mathcal{L}_{1,42} = \{B_3 + \lambda P_3\}$
$\mathcal{L}_{1,7} = \{B_1 + B_5 + aP_4 + \epsilon P_5\}$	$\mathcal{L}_{1,25} = \{B_1 - B_2\}$	$\mathcal{L}_{1,43} = \{B_3 + \lambda P_3 + aP_4\}$
$\mathcal{L}_{1,8} = \{B_1 + B_5 + \epsilon P_4\}$	$\mathcal{L}_{1,26} = \{B_1 - B_2 + \delta L + aD_1\}$	$\mathcal{L}_{1,44} = \{B_3 + \lambda P_4\}$
$\mathcal{L}_{1,9} = \{B_1 + P_3 + P_5\}$	$\mathcal{L}_{1,27} = \{B_1 - B_2 + \lambda L + aD_1 + \delta P_5\}$	$\mathcal{L}_{1,45} = \{B_3 + P_5\}$
$\mathcal{L}_{1,10} = \{B_1 + P_5\}$	$\mathcal{L}_{1,28} = \{B_1 - B_2 + \lambda D_1\}$	$\mathcal{L}_{1,46} = \{B_5\}$
$\mathcal{L}_{1,11} = \{D_2\}$	$\mathcal{L}_{1,29} = \{B_1 - B_2 + \lambda D_1 + \delta P_5\}$	$\mathcal{L}_{1,47} = \{B_5 + P_1 + aP_2\}$
$\mathcal{L}_{1,12} = \{D_2 + \lambda L + aD_1\}$	$\mathcal{L}_{1,30} = \{B_1 - B_2 + \lambda P_5\}$	$\mathcal{L}_{1,48} = \{B_5 + P_2 + aP_4\}$
$\mathcal{L}_{1,13} = \{D_2 + \lambda L + aD_1 + \delta P_5\}$	$\mathcal{L}_{1,31} = \{L + aD_1\}$	$\mathcal{L}_{1,49} = \{B_5 + P_5\}$
$\mathcal{L}_{1,14} = \{D_2 + \lambda D_1\}$	$\mathcal{L}_{1,32} = \{L + aD_1 + \delta P_5\}$	$\mathcal{L}_{1,50} = \{P_1\}$
$\mathcal{L}_{1,15} = \{D_2 + \lambda D_1 + \delta P_5\}$	$\mathcal{L}_{1,33} = \{D_1\}$	$\mathcal{L}_{1,51} = \{P_1 + P_3\}$
$\mathcal{L}_{1,16} = \{D_2 + \delta P_5\}$	$\mathcal{L}_{1,34} = \{D_1 + \delta P_5\}$	$\mathcal{L}_{1,52} = \{P_3\}$
$\mathcal{L}_{1,17} = \{D_2 - D_1 + B_3\}$	$\mathcal{L}_{1,34} = \{B_3\}$	$\mathcal{L}_{1,53} = \{P_5\}$
$\mathcal{L}_{1,18} = \{D_2 - D_1 + B_3 + \delta P_2\}$	$\mathcal{L}_{1,36} = \{B_3 + B_5\}$	

Table 3. List of 1-dimensional representative subalgebras of $\mathcal{L}$. The parameters are $\epsilon = \pm 1$ and $a, \lambda, \delta, \in \mathbb{R}$, where $\delta \neq 0$, $\lambda > 0$.

2.2. Classification into conjugacy classes of the subalgebra $\mathcal{S}$.

For the sake of classification, we decompose the seven-dimensional subalgebra $\mathcal{S}$ into the direct sum $\mathcal{G} \oplus \{D_1\}$, with $\mathcal{G} = \{B_1, D_2, B_2, K, L, P_5\}$, which is further decomposed as follows:

$$\mathcal{G} = \mathcal{A} \triangleright \mathcal{M},$$

where $\mathcal{A}$ is the simple algebra of the previous subsection and $\mathcal{M} = \{K, L, P_5\}$ is the ideal. Applying the method [11, 12, 13], we proceed to classify all subalgebras of $\mathcal{L}$ into conjugacy classes under the action of the automorphisms generated by G and the discrete transformations (10). In practice, we can classify the subalgebras under the automorphisms generated by G and decrease the range of the parameters that appear in the representative subalgebra of a class using the Lie algebra automorphisms (11). The classification results for $\mathcal{S}$ are shown in table 4 for subalgebras of codimension 1 and in table 5 for subalgebras of codimension 2.

$\mathcal{S}_{1,1} = \{B_1\}$	$\mathcal{S}_{1,14} = \{B_1 - B_2 + \delta L\}$	$\mathcal{S}_{1,27} = \{D_2 + \delta_1 L + \delta_2 D_1\}$
$\mathcal{S}_{1,2} = \{B_1 + K\}$	$\mathcal{S}_{1,15} = \{B_1 - B_2 + L + \epsilon P_5\}$	$\mathcal{S}_{1,28} = \{D_2 + \delta P_5\}$
$\mathcal{S}_{1,3} = \{B_1 + K + \delta D_1\}$	$\mathcal{S}_{1,16} = \{B_1 - B_2 + L + \epsilon_1 P_5 + \epsilon_2 D_1\}$	$\mathcal{S}_{1,29} = \{D_2 + \delta_1 L + \delta_2 D_1\}$
$\mathcal{S}_{1,4} = \{B_1 + L\}$	$\mathcal{S}_{1,17} = \{B_1 - B_2 + \delta_1 L + \delta_2 D_1\}$	$\mathcal{S}_{1,30} = \{D_2 + \delta D_1\}$
$\mathcal{S}_{1,5} = \{B_1 + L + \epsilon P_5\}$	$\mathcal{S}_{1,18} = \{B_1 - B_2 + \delta P_5\}$	$\mathcal{S}_{1,31} = \{K\}$
$\mathcal{S}_{1,6} = \{B_1 + L + \epsilon_1 D_1 + \epsilon_2 P_5\}$	$\mathcal{S}_{1,19} = \{B_1 - B_2 + \delta_1 P_5 + \delta_2 D_1\}$	$\mathcal{S}_{1,32} = \{K + \delta D_1\}$
$\mathcal{S}_{1,7} = \{B_1 + L + \delta D_1\}$	$\mathcal{S}_{1,20} = \{B_1 - B_2 + \delta D_1\}$	$\mathcal{S}_{1,33} = \{L\}$
$\mathcal{S}_{1,8} = \{B_1 + P_5\}$	$\mathcal{S}_{1,21} = \{D_2\}$	$\mathcal{S}_{1,34} = \{L + \delta D_1\}$
$\mathcal{S}_{1,9} = \{B_1 + P_5 + \delta D_1\}$	$\mathcal{S}_{1,22} = \{D_2 + \delta K\}$	$\mathcal{S}_{1,35} = \{L + \epsilon P_5\}$
$\mathcal{S}_{1,10} = \{B_1 + D_1\}$	$\mathcal{S}_{1,23} = \{D_2 + \delta_1 K + \delta_2 D_1\}$	$\mathcal{S}_{1,36} = \{L + \epsilon_1 P_5 + \epsilon_2 D_1\}$
$\mathcal{S}_{1,11} = \{B_1 - B_2\}$	$\mathcal{S}_{1,24} = \{D_2 + \delta L\}$	$\mathcal{S}_{1,37} = \{P_5\}$
$\mathcal{S}_{1,12} = \{B_1 - B_2 + \delta K\}$	$\mathcal{S}_{1,25} = \{D_2 + L + \epsilon_1 P_5\}$	$\mathcal{S}_{1,38} = \{P_5 + \delta D_1\}$
$\mathcal{S}_{1,13} = \{B_1 - B_2 + \delta_1 K + \delta_2 D_1\}$	$\mathcal{S}_{1,26} = \{D_2 + L + \epsilon_1 P_5 + \epsilon_2 D_1\}$	$\mathcal{S}_{1,39} = \{D_1\}$

Table 4. List of 1-dimensional representative subalgebras of $\mathcal{S}$. The parameters are $\epsilon, \epsilon_1, \epsilon_2 = \pm 1$ and $\delta, \delta_1, \delta_2 \in \mathbb{R}$, $\delta, \delta_1, \delta_2 \neq 0$.

3. Invariant and partially invariant solutions.

Since equations (1.a) and (1.b) do not involve the velocity components u and v, they can first be solved for θ and σ. Next, the result is introduced into the system formed by equations (1.c) and (1.d) and we look for the solution of this system for the velocity components u and v. This system always admits the particular solution

$$u = b_1 y + b_2, \qquad v = -b_1 x + b_3, \quad b_1, b_2, b_3 \in \mathbb{R}, \tag{15}$$

obtained by requiring that the coefficients of the trigonometric functions in (1.c) vanish. The velocity field defined by (15) forms concentric circles. Therefore, this solution does not establish any relation between the flow velocity and the strain involved in the plastic material. Therefore, it is not an interesting result by itself from the physical point of view. Nevertheless, since the PDEs (1.c) and (1.d) are linear (assuming that θ is known), they admit a linear superposition principle. Then, we can add the solution (15) to any

$\mathcal{S}_{2,1} = \{B_1, D_2\}$	$\mathcal{S}_{2,32} = \{B_1 - B_2 + \delta K, D_1\}$
$\mathcal{S}_{2,2} = \{B_1, D_2 + \delta K\}$	$\mathcal{S}_{2,33} = \{B_1 - B_2 + \delta L, D_1\}$
$\mathcal{S}_{2,3} = \{B_1, D_2 + \delta L\}$	$\mathcal{S}_{2,34} = \{B_1 - B_2 + L + \epsilon P_5, D_1\}$
$\mathcal{S}_{2,4} = \{B_1, D_2 + \epsilon_1 L + \epsilon_2 P_5\}$	$\mathcal{S}_{2,35} = \{B_1 - B_2 + \delta P_5, D_1\}$
$\mathcal{S}_{2,5} = \{B_1, D_2 + \delta P_5\}$	$\mathcal{S}_{2,36} = \{B_1 - B_2 + \delta D_1, K + aD_1\}$
$\mathcal{S}_{2,6} = \{B_1, D_2 + \delta_1 K + \delta_2 D_1\}$	$\mathcal{S}_{2,37} = \{B_1 - B_2 + \delta D_1, L + a_1 D_1\}$
$\mathcal{S}_{2,7} = \{B_1, D_2 + \epsilon_1 L + \epsilon_2 P_5\}$	$\mathcal{S}_{2,38} = \{B_1 - B_2 + \delta D_1, L + \epsilon_1 P_5 + \epsilon_2 D_1\}$
$\mathcal{S}_{2,8} = \{B_1, D_2 + \epsilon_1 L + \epsilon_2 P_5 + \epsilon_3 D_1\}$	$\mathcal{S}_{2,39} = \{B_1 - B_2 + \delta D_1, P_5 + a_1 D_1\}$
$\mathcal{S}_{2,9} = \{B_1, D_2 + \delta_1 L + \delta_2 D_1\}$	$\mathcal{S}_{2,40} = \{D_2, K\}$
$\mathcal{S}_{2,10} = \{B_1, D_2 + \epsilon_1 L + \epsilon_2 P_5 + \epsilon_3 D_1\}$	$\mathcal{S}_{2,41} = \{D_2, L\}$
$\mathcal{S}_{2,11} = \{B_1, D_2 + \delta_1 P_5 + \delta_2 D_1\}$	$\mathcal{S}_{2,42} = \{D_2, L + \epsilon P_5\}$
$\mathcal{S}_{2,12} = \{B_1, D_2 + \delta D_1\}$	$\mathcal{S}_{2,43} = \{D_2, P_5\}$
$\mathcal{S}_{2,13} = \{B_1, K\}$	$\mathcal{S}_{2,44} = \{D_2, D_1\}$
$\mathcal{S}_{2,14} = \{B_1, L\}$	$\mathcal{S}_{2,45} = \{D_2 + \delta K, D_1\}$
$\mathcal{S}_{2,15} = \{B_1, L + \epsilon P_5\}$	$\mathcal{S}_{2,46} = \{D_2 + \delta L, D_1\}$
$\mathcal{S}_{2,16} = \{B_1, P_5\}$	$\mathcal{S}_{2,47} = \{D_2 + L + \epsilon P_5, D_1\}$
$\mathcal{S}_{2,17} = \{B_1, D_1\}$	$\mathcal{S}_{2,48} = \{D_2 + \delta P_5, D_1\}$
$\mathcal{S}_{2,18} = \{B_1 + K, D_1\}$	$\mathcal{S}_{2,49} = \{D_2 + \lambda D_1, K + aD_1\}$
$\mathcal{S}_{2,19} = \{B_1 + L, D_1\}$	$\mathcal{S}_{2,50} = \{D_2 + \lambda D_1, L + aD_1\}$
$\mathcal{S}_{2,20} = \{B_1 + L + \epsilon P_5, D_2 + 2\epsilon_2 K\}$	$\mathcal{S}_{2,51} = \{D_2 + \lambda D_1, L + \epsilon_1 P_5 + \epsilon_2 D_1\}$
$\mathcal{S}_{2,21} = \{B_1 + L + \epsilon P_5, D_1\}$	$\mathcal{S}_{2,52} = \{D_2 + \lambda D_1, P_5 + aD_1\}$
$\mathcal{S}_{2,22} = \{B_1 + P_5, D_1\}$	$\mathcal{S}_{2,53} = \{L, P_5 + aD_1\}$
$\mathcal{S}_{2,23} = \{B_1 + D_1, K + aD_1\}$	$\mathcal{S}_{2,54} = \{L + \delta D_1, P_5\}$
$\mathcal{S}_{2,24} = \{B_1 + D_1, L + aD_1\}$	$\mathcal{S}_{2,55} = \{L + \epsilon_1 D_1, P_5 + \epsilon_2 D_1\}$
$\mathcal{S}_{2,25} = \{B_1 + D_1, L + \epsilon_1 P_5 + \epsilon_2 D_1\}$	$\mathcal{S}_{2,56} = \{K, L + \epsilon P_5\}$
$\mathcal{S}_{2,26} = \{B_1 + D_1, P_5 + aD_1\}$	$\mathcal{S}_{2,57} = \{L, P_5\}$
$\mathcal{S}_{2,27} = \{B_1 - B_2, K\}$	$\mathcal{S}_{2,58} = \{K, D_1\}$
$\mathcal{S}_{2,28} = \{B_1 - B_2, L\}$	$\mathcal{S}_{2,59} = \{L, D_1\}$
$\mathcal{S}_{2,29} = \{B_1 - B_2, L + \epsilon P_5\}$	$\mathcal{S}_{2,60} = \{L + \epsilon P_5, D_1\}$
$\mathcal{S}_{2,30} = \{B_1 - B_2, P_5\}$	$\mathcal{S}_{2,61} = \{P_5, D_1\}$
$\mathcal{S}_{2,31} = \{B_1 - B_2, D_1\}$	

Table 5. List of 2-dimensional representative subalgebras of $\mathcal{S}$. The parameters are $\epsilon, \epsilon_1, \epsilon_2 = \pm 1$ and $a, \lambda, \delta, \delta_1, \delta_2 \in \mathbb{R}$, $\delta, \delta_1, \delta_2 \neq 0$, $\lambda > 0$.

solution, for the velocity components, of the system (1.c), (1.d), corresponding to given solutions θ and σ of the system (1.a), (1.b). Consequently, we can satisfy a much broader family of boundary conditions.

3.1. Symmetry reduction for the representative subalgebra B_1

Consider for illustration the one-dimensional representative subalgebra generated by the infinitesimal generator

$$B_1 = -v\partial_x + u\partial_y. \tag{16}$$

Since no derivative with respect to variables u, v, θ, σ appears in B_1, it follows that these variables are all invariants of the subalgebra generated by B_1. In order to obtain a complete set of functionally independent invariants, one can include also the symmetry variable

$$\xi = ux - vy. \tag{17}$$

We look for a solution of the form

$$u = F(\xi), \quad v = G(\xi), \quad \theta = T(\xi), \quad \sigma = S(\xi), \tag{18}$$

where ξ is defined by (17). Replacing (18) into the original system (16) and assuming that $1 - xF'(\xi) - yG'(\xi) \neq 0$, where $F'(\xi) = dF(\xi)/d\xi$, etc., so that we can use the Implicit Function Theorem, we obtain a system of equations with a reduced number of independent variables, where the functions F, G, T, S are to be determined. This system takes the form of four coupled ODEs:

$$\begin{aligned}
&F(\xi)S'(\xi) - (\cos(2T(\xi))F(\xi) + \sin(2T(\xi))G(\xi))\,T'(\xi) = 0,\\
&G(\xi)S'(\xi) - (\sin(2T(\xi))F(\xi) - \cos(2T(\xi))G(\xi))\,T'(\xi) = 0,\\
&(G(\xi)F'(\xi) + F(\xi)G'(\xi))\sin(2T(\xi)) + (F(\xi)F'(\xi) - G(\xi)G'(\xi))\cos(2T(\xi)) = 0,\\
&F(\xi)F'(\xi) + G(\xi)G'(\xi) = 0.
\end{aligned} \tag{19}$$

The solution of this system is

$$F(\xi) = c_1\cos(T(\xi)), \qquad G(\xi) = c_1\sin(T(\xi)), \quad S(\xi) = T(\xi) + c_2, \tag{20}$$

where $T(\xi)$ is an arbitrary function of a single variable and c_1, c_2 are constants of integration. The solution is obtained by replacing expressions (20) for F, G, T, S into equation (18). The solution of the original system is given implicitly by relations

$$u = c_1\cos(ux + yv), \qquad v = c_1\sin(ux + yv), \tag{21}$$

while the angle θ and the pressure σ are defined by the choice of the arbitrary function T as follows:

$$\theta = T(ux + yv), \qquad \sigma = T(ux + yv) + c_2. \tag{22}$$

Since by defining θ through a certain choice of T, we also determine σ, it follows that equation (22) is a relation defining the pressure σ in terms of the angle θ or vice-versa. Moreover, we can see from equation (21) that the sum of squares of the velocity components u and v, is constant. Therefore, since the material is incompressible, the solution preserves the kinetic energy of the plastic material, *i.e.*

$$u^2 + v^2 = c_1^2.$$

For the purpose of illustration, consider the function:

$$T(\xi) = (1/2)\arcsin(\xi), \tag{23}$$

This particular choice of T allows us to solve relations (21) in order to find the velocities u and v explicitly in terms of x and y. The obtained formulas can be expressed in terms of radicals and are very involved. Therefore, they are omitted here. Nevertheless, these formulas can be used to trace the vector fields corresponding to solution (21), where T is defined by (23). An example of such a tool is presented in figure 1 for a flow velocity $c_1 = 5$. The feeding velocity used is $(U_0, V_0) = (4.30, 2.55)$ and the extraction speed is $(U_1, V_1) = (-4.30, 2.55)$. The boundaries of the extrusion die are chosen in such a way that they coincide with the flow lines of the velocity field. Therefore, they are solutions of the equation

$$dy/dx = v(x, y)/u(x, y).$$

For figure 1, the inner boundary corresponds to the initial value $(x_0, y_0) = (-0.5, -0.35)$ and the outer boundary to the initial value $(x_0, y_0) = (-0.43, -0.46)$. The curves $\mathcal{C}_1$ and $\mathcal{C}_2$ are the limits of the plasticity region with respect to the entrance and exit of the extrusion die. They are solutions of equation (2), where, (U_0, V_0) is replaced by (U_1, V_1) for $\mathcal{C}_2$. In order to define the limit of the plasticity region at the ends of the boundary of the tool, the initial data used to trace the curve $\mathcal{C}_1$ are $(x_0, y_0) = (-0.5, -0.35)$ while for $\mathcal{C}_2$ they are $(x_0, y_0) = (-0.5, 0.35)$. Numerical integration has been used to identify the boundary of the tool and the limits of the region of plasticity. This type of extrusion die can be used to bend a rectangular rod or a slab of a ideal plastic material. The average pressure and the angle θ, which define the strain tensor inside the tool, are evaluated by formulas (22), where T is defined by (23).

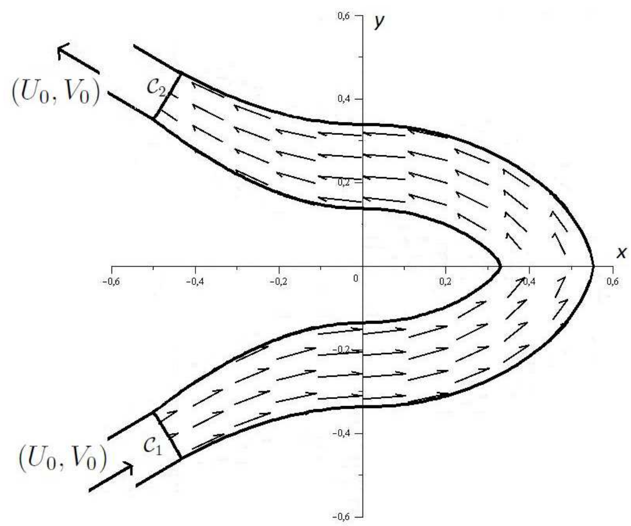

Figure 1. Extrusion die corresponding to the solution (21).

3.2. Symmetry reduction for the representative subalgebra K

As an example, we find a partially invariant solution corresponding to the subalgebra generated by generator K which admits the following invariants:

$$\xi = xy\cos(2\theta) - (1/2)(x^2 - y^2)\sin(2\theta), \quad F = xu + yv. \tag{24}$$

and

$$S = \theta^2 - \sigma^2, \quad G = uv\cos(2\theta) - (1/2)(u^2 - v^2)\sin(2\theta).$$

In order to obtain a PIS, we use only the two invariants given by (24). We begin by inverting the first relation in (24) in order to find θ as a function of ξ. Next, we introduce the result in the first two equations of system (1). Then, comparing the values of the mixed derivatives of $\sigma(x, y)$ with respect to x and y, we obtain the following PDE for

the quantity ξ

$$\begin{aligned}&(\xi_{xx}-\xi_{yy})\left(((x^2+y^2)^2-4\xi^2)(\xi(x^2-y^2)-xy\sqrt{(x^2+y^2)^2-4\xi^2})\right)\\&-\frac{4xy\xi+(x^2-y^2)\sqrt{(x^2+y^2)^2-4\xi^2}}{xy\sqrt{(x^2+y^2)^2-4\xi^2}-(x^2-y^2)\xi}\xi_{xy}+(x^2+y^2)^2\left((x+y)\xi_x-(x-y)\xi_y\right)\\&\times\left((x-y)\xi_x+(x+y)\xi_y\right)-4(x^2+y^2)^2\xi\left(x\xi_x+y\xi_y-\xi\right)=0\end{aligned}\tag{25}$$

There are two particular solutions for ξ to equation (25) defined by

$$\xi=\frac{1}{2}(x^2+\epsilon y^2),\qquad \epsilon=\pm 1.\tag{26}$$

Let us consider the case $\epsilon=1$ and introduce this solution into the first relation (24). Solving for θ, we obtain:

$$\theta=-\frac{1}{2}\arctan\left(\frac{x^2-y^2}{xy}\right).\tag{27}$$

The mean pressure σ is found by quadrature from the first two equations (1) in which we have introduced the solution (27) for θ. The result for σ is:

$$\sigma=-(1/2)\ln\left(x^2+y^2\right)+c_1,\tag{28}$$

where c_1 is a real integration constant.

Using the form of the second invariant in (24), we look for a solution for the components u and v of the velocity, with the form

$$u=(y/x)v-F(\xi),\tag{29}$$

where the symmetry variable ξ is given by (26). By replacing θ given by (27) and u by (29) into the system composed of the last two equations in (1), then using the compatibility condition of the mixed derivatives of v with respect to x and y, we obtain the condition that $F=c_2$, where c_2 is a real constant. The solution for u and v is then:

$$u=\frac{c_2x}{x^2+y^2}+c_3y+c_4,\quad v=\frac{c_2y}{x^2+y^2}-c_3x+c_5,\tag{30}$$

where $c_3\ldots,c_5$ are real constants of integration. Note that in the case when $c_4=c_5=0$ and $c_2\neq 0\neq c_3$, the flow lines form logarithmic spirals centered at the origin.

An example of velocity fields is given in Figure 2 for parameters $c_2=-1$, $c_3=-2$, $c_4=4$ and $c_5=1$ for solution (30). The chosen region, $[-1,1]\times[-1,1]$, includes the singularity at the origin. Corresponding to this solution for the same parameters, an extrusion tool is given in Figure 3 for the feeding and extraction velocities $(U_0,V_0)=(5.5,0)$ and $(U_1,V_1)=(3,3)$ respectively. The curve $\mathcal{C}_1$ is the limit of the plasticity region at the entrance of the extrusion die and $\mathcal{C}_2$ has the same significance at the exit of the extrusion die. The upper contour of the extrusion die is a solution of $dy/dx=v/u$, with u and v defined by (30), for an initial value $y(-0.5)=-0.8$ while, for the lower contour, we have used the initial value $y(-0.7)=-0.95$.

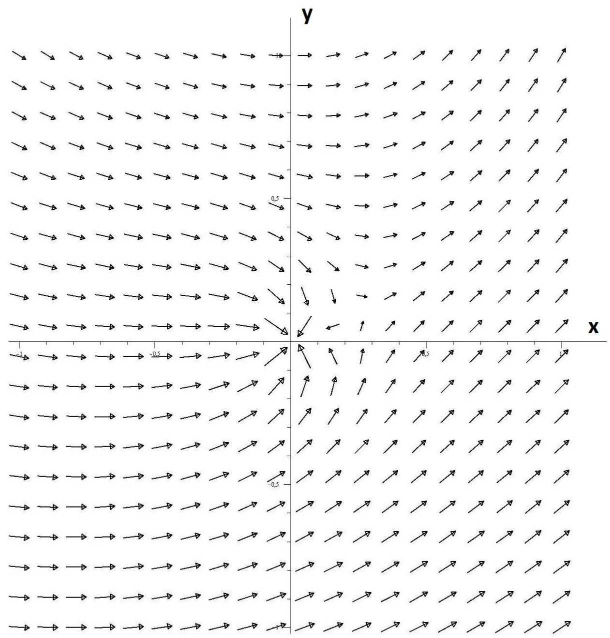

Figure 2. Extrusion die corresponding to the solution (30).

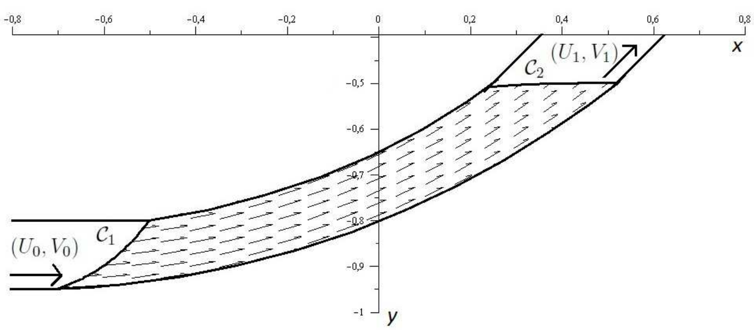

Figure 3. Extrusion die corresponding to the solution (30).

3.3. Similarity solution for the angle θ and corresponding pressure σ

In this section, we find solutions of the system (1) for which the angle θ is a similarity solution. We propose the solution for θ in the form

$$\theta(x, y) = J(\xi(x, y)), \tag{31}$$

where the symmetry variable is of the form

$$\xi(x, y) = y/x. \tag{32}$$

One should note that the solutions, obtained by assuming the hypotheses (31) and (32) on their form, are more general than the invariant solutions corresponding to subalgebras

which admit y/x as a symmetry variable. This is so because the invariance requirement, for a given subalgebra, leads to constraints on the form of σ, which is not the case here. The introduction of (31), with ξ defined by (32), in the system (1.a), (1.b), leads to the system

$$\begin{aligned}\sigma_x(x,y) &= 2(k/x)\left(-\xi(x,y)\cos(J(\xi(x,y))) + \sin(\xi(x,y))\right) J'(\xi(x,y)),\\ \sigma_y(x,y) &= -2(k/x)\left(\xi(x,y)\sin(J(\xi(x,y))) + \cos(J(\xi(x,y)))\right) J'(\xi(x,y)).\end{aligned} \tag{33}$$

Considering the compatibility condition on mixed derivatives of σ relative to x and y, we deduce from (33) the following ODE for the function J:

$$\begin{aligned}&\left((\xi^2-1)\sin(2J(\xi)) + 2\xi\cos(2J(\xi))\right) J''(\xi) + 2\left(\xi\sin(2J(\xi)) + \cos(J(\xi))\right) J'(\xi)\\ &+ 2\left(-2\sin(2J(\xi)) + (\xi^2-1)\cos(2J(\xi))\right)(J'(\xi))^2 = 0,\end{aligned} \tag{34}$$

which has the first integral

$$\left((\xi^2-1)\sin(2J(\xi)) + 2\xi\cos(2J(\xi))\right) J'(\xi) = c_1, \tag{35}$$

where c_1 is an integration constant. There are two cases to consider to solve the equation (35).

i). If $c_1 \neq 0$, the solution of (35) is given in implicit form by

$$\frac{(\tan(J(\xi)) - \xi)\sqrt{c_1^2-1}}{(\tan(J(\xi))\xi + 1)\, c_1 - \xi + \tan(J(\xi))} - \tan\left(\frac{\sqrt{c_1^2-1}(c_2 - J(\xi))}{c_1}\right) = 0, \tag{36}$$

where c_2 is an integration constant. The solution for σ is obtained by integrating the system (33) and taking into account the first integral (35). We find

$$\sigma(x,y) = k\left(\xi(x,y)\cos(2J(\xi(x,y))) - \sin(2J(\xi(x,y))) - 2c_1\ln(x)\right) + c_3, \quad c_3 \in \mathbb{R} \tag{37}$$

where $\xi(x,y)$ is defined by (32) and $J(\xi)$ is a solution of (36). So, $\theta(x,y) = J(y/x)$ and $\sigma(x,y)$ are given by (37) and are solutions of the system (1.a), (1.b), if the function $J(\xi)$ satisfies the algebraic equation (36).

ii). If the constant $c_1 = 0$ in equation (36), then the solution of (34) for J is

$$J(\xi) = -\frac{1}{2}\arctan\left(\frac{2\xi}{\xi^2-1}\right). \tag{38}$$

We subsequently solve (33) for $\sigma(x,y)$ considering (38), which leads to the solution of the system (1.a), (1.b), given by

$$\theta(x,y) = \frac{1}{2}\arctan\left(\frac{2xy}{x^2-y^2}\right), \qquad \sigma(x,y) = -2k\arctan\left(\frac{y}{x}\right) + c_2. \tag{39}$$

3.3.1. Additive separation for the velocities when $c_1 \neq 0$. Already knowing the solution $\theta(x,y)$, $\sigma(x,y)$, in the case where $c_1 \neq 0$, we still have to compute the solution for u and v. One way to proceed is to suppose that the solution is in the additive separated form

$$u(x,y) = f(x,y) + F(\xi(x,y)), \qquad v(x,y) = g(x,y) + G(\xi(x,y)), \tag{40}$$

where $\xi(x,y) = y/x$. We introduce (40) into the system (1.c), (1.d), which gives

$$\begin{aligned}&(\sin(2J(\xi)) - \cos(2J(\xi)))\,\xi F'(\xi) - (\cos(2J(\xi)) + \sin(2J(\xi)))\,\xi G'(\xi)\\ &+ ((f_y + g_x)\sin(2J(\xi)) + (f_x - g_y)\cos(2J(\xi)))\,x = 0,\end{aligned} \tag{41}$$

$$(f_x + g_y)\,x + G'(\xi) - \xi F'(\xi) = 0, \tag{42}$$

We must determine which functions f and g will reduce equations (41), (42), to a system of ODEs for the single-variable functions $F(\xi)$ and $G(\xi)$. To reach this goal, we first use as annihilator the infinitesimal generator $(1/2)(D_1 + D_2)$ defined by (4) that we apply to the equations (41), (42), to eliminate the presence of the functions F and G. Indeed, the operator $(1/2)(D_1 + D_2)$ annihilates any function of $\xi = y/x$. So, we obtain as differential consequences some conditions on the functions f and g. We can assume that $f_x(x,y) + g_y(x,y) \neq 0$, otherwise we can show that the only possible solution is the trivial constant solution for u and v. Under this hypothesis, the previous conditions read

$$f_x = -g_y + \zeta_1(\xi)x^{-1}, \tag{43}$$

$$f_y = -g_x + g_y\zeta_2(\xi) + \zeta_3(\xi)x^{-1}, \tag{44}$$

$$(g_y + (D_1 + D_2)(g_y))\,(\zeta_2(\xi)\sin(2J(\xi)) - 2\cos(2J(\xi))) = 0, \tag{45}$$

where the functions of one variable ζ_i, $i = 1,2,3$ are arbitrary. Since the left member of (45) is composed of two factors, we must consider two possibilities.

(a). We first suppose that

$$g_y + (D_1 + D_2)(g_y) = 0. \tag{46}$$

In this case, we find that the functions f and g take the form

$$\begin{aligned}f(x,y) =& -\int^{\xi(x,y)} \frac{\zeta_1(\xi) - \zeta_4'(\xi)}{\xi}d\xi + \omega_4\ln(y) - \omega_1 y + \omega_5,\\ g(x,y) =& \zeta_4(\xi) + \omega_2\ln(x) + \omega_1 x + \omega_3,\end{aligned} \tag{47}$$

where the functions $\zeta_1(\xi), \zeta_4(\xi)$ are arbitrary and the functions $\zeta_2(\xi)$, $\zeta_3(\xi)$, were chosen to solve the compatibility conditions on mixed derivatives of f relative to x and y. We now introduce the solution (47) in the system (43), (44), which leads to an ODE system

for F and G, that we omit due to its complexity, and for which the solutions take the form of quadratures

$$
\begin{aligned}
F(\xi) &= \int \left(-\zeta_4'(\xi) + \frac{\zeta_1(\xi)}{\xi} + \frac{(\omega_4 + \omega_2 \xi)\sin(2J(\xi))}{\xi\left((\xi^2-1)\sin(2J(\xi)) + 2\xi\cos(2J(\xi))\right)} \right) d\xi + c_5, \\
G(\xi) &= \int \left(-\zeta_1(\xi) + \xi F'(\xi)\right) d\xi + c_4, \qquad c_4, c_5 \in \mathbb{R}.
\end{aligned}
\tag{48}
$$

The last step is to introduce (47) and (48) in the Ansatz (40). Then, the velocities are

$$
\begin{aligned}
u(x,y) =& -\frac{1}{2}\frac{c_5\cos(2J(y/x))}{c_1} + \int^{y/x} \frac{c_6\sin(2J(\xi))J'(\xi)}{c_1\xi} d\xi + c_6\ln(y) - c_4 y + c_7, \\
v(x,y) =& c_5\ln(x) + c_4 x + \int^{y/x} c_1^{-1}c_5\xi\sin(2J(\xi))J'(\xi)d\xi - (2c_1)^{-1}(c_6\cos(2J(\xi))) + c_8,
\end{aligned}
\tag{49}
$$

where the c_i are integration constants. So, we obtain a solution of the system (1) by defining the angle θ by (31), (36), the mean pressure σ by (37) and the velocities u and v by (49), with $\xi(x,y) = y/x$.

(b). Suppose now that the condition (45) is satisfied by requiring

$$
\zeta_2(\xi) = 2\cot(2J(\xi)). \tag{50}
$$

Then, applying the compatibility condition on mixed derivative of f relative to x and y to the equations (43), (44), and considering ζ_2 given by (50), we conclude that the function g must solve the equation

$$
\begin{aligned}
& g_{xx}(x,y) + 2\cot(2J(\xi))g_{xy}(x,y) - g_{yy}(x,y) + 4x^{-1}\xi\left(\xi + \cot^2(2J(\xi))\right)J'(\xi)g_y(x,y) \\
& + x^{-2}\left(\zeta_3'(\xi)\xi + \zeta_1'(\xi) + \zeta_3(\xi)\right) = 0.
\end{aligned}
\tag{51}
$$

It's a hyperbolic equation everywhere in the domain where J is defined. So, we introduce the change of variable

$$
\begin{aligned}
\phi(x,y) =& x\exp\left(\int^{\xi(x,y)} \frac{\sin(2J(\xi))}{1 + \cos(2J(\xi)) + \xi\sin(2J(\xi))} d\xi\right), \\
\psi(x,y) =& x\exp\left(\int^{\xi(x,y)} \frac{\sin(2J(\xi))}{-1 + \cos(2J(\xi)) + \xi\sin(2J(\xi))} d\xi\right),
\end{aligned}
\tag{52}
$$

which brings the equation (51) to the simplified form

$$
\begin{aligned}
& g_{\phi,\psi} + \frac{c_1}{2}\left(\frac{\sin(2J(\phi,\psi))g_\phi}{\psi(\cos(2J(\phi,\psi))) + 1} - \frac{\sin(2J(\phi,\psi))g_\psi}{\phi(\cos(2J(\phi,\psi)) - 1)}\right) \\
& - \frac{1}{4}\frac{\sin(2J(\phi,\psi))\xi^2 - \sin(2J(\phi,\psi)) + 2\xi\cos(2J(\phi,\psi))\left(\zeta_3(\xi) + \zeta_1'(\xi) + \xi\zeta_3'(\xi)\right)}{J(\phi,\psi)} = 0,
\end{aligned}
\tag{53}
$$

where $J(\phi, \psi)$ is defined by

$$J = (c_1/4)(\psi - \phi). \tag{54}$$

To solve the equation (53) more easily, we define the function ζ_3 by

$$\zeta_3(\xi) = \xi^{-1}\left(-\zeta_1(\xi) + J(\xi) + \omega_1\right). \tag{55}$$

So, the solution of (53) is

$$g(\phi,\psi) = -(1/2)\left(\omega_1 - (1/2)\ln(\psi/\phi)\right)\cos\left(c_1\ln(\psi/\psi)\right) - (1/4)c_1^{-1}\sin\left(c_1\ln(\psi/\phi)\right), \tag{56}$$

which, returning to the initial variables, takes the form

$$g(x,y) = -(1/2)\big(\omega_1 - (1/2)c_1^{-1}J(y/x)\big)\cos(2J(y/x)) - (1/4)c_1^{-1}\sin(2J(y/x)) + \omega_2. \tag{57}$$

After the introduction of the solution (57) for g, the function f is given by quadrature from the equations (43), (44). The obtained solution for f is

$$f(x,y) = \left(\int^{\xi(x,y)} \frac{(c_1\omega_1 - J(\xi))\sin(2J(\xi))J'(\xi)}{c_1\xi}d\xi\right) - \int^{\xi(x,y)} \frac{\zeta_1(\xi)}{\xi}d\xi + (c_1+1)\omega_1\ln(y) + \omega_3. \tag{58}$$

We now introduce (57), (58), in (41), (42), and get F and G by quadrature in the form

$$\begin{aligned} F(\xi) &= \int^{\xi(x,y)} \frac{\zeta_1(\xi)}{\xi}d\xi + \int^{\xi(x,y)} \frac{(\omega_1 + J(\xi))\sin(2J(\xi))J'(\xi)}{c_1\xi}d\xi, \\ G(\xi) &= -(1/2)c_1^{-1}\left((\omega_1 + J(\xi))\cos(2J(\xi)) + 2\sin(2J(\xi))\right). \end{aligned} \tag{59}$$

Finally, the substitution of (57), (58) and (59) in (40) provides the solution to (1.c), (1.d):

$$\begin{aligned} u(x,y) =& (c_1+1)c_4\ln(y) + \int^{y/x} \frac{c_4(c_1+1)\sin(2J(\xi))J'(\xi)}{c_1\xi}d\xi + c_5, \\ v(x,y) =& -\frac{c_4(c_1+1)\cos(2J(y/x))}{c_1} + c_6, \end{aligned} \tag{60}$$

where the c_i are integration constants. So, we have a solution of the system (1) by implicitly defining the angle θ by (31), (36), the mean pressure by σ by (37) and the velocities u and v by (60), with $\xi(x,y) = y/x$.

3.3.2. Additive separation for the velocities u and v when $c_1 = 0$. Now, we consider the case where $c_1 = 0$ in (35). Then the solutions for θ and σ are given by (39). We still suppose that the solution for u and v is in the form (40). The procedure is the same as for the previous case until we obtain the conditions (43), (44) and (45). We must again consider two distinct cases.

(a.) We first suppose that the condition (46) is satisfied. Then, the functions f and g are defined by

$$\begin{aligned} f(x,y) =& \int^{\xi(x,y)} (\zeta_2(\xi)+\xi)\zeta_4'(\xi) - \zeta_3(\xi) d\xi - \omega_1 y + \omega_3, \\ g(x,y) =& \zeta_4(\xi(x,y)) + \omega_1 x + \omega_2, \qquad \omega_i \in \mathbb{R},\ i=1,2,3, \end{aligned} \tag{61}$$

where the $\zeta_i, = 1,2,3,$ are arbitrary functions of one variable and, to simplify the expression for f and g, we have chosen $\zeta_1(\xi) = (1 - \zeta_2(\xi)\xi - \xi^2)\,\zeta_4(\xi) - \xi\zeta_3(\xi)$. We substitute (61) in equations (41) and (42) to determine F and G. We conclude that $F(\xi)$ is an arbitrary function, while $G(\xi)$ is expressed as the quadrature

$$G(\xi) = \int \left(F'(\xi)\xi + \left(\xi\zeta_2(\xi) - 1 + \xi^2\right)\zeta_4' - \xi\zeta_3(\xi)\right) d\xi. \tag{62}$$

We finally obtain the solution u and v by introducing (61), (62) in (40) and, to simplify, by choosing

$$\zeta_3(\xi) = -\left(\zeta_2(\xi) + \xi\right)\zeta_4'(\xi),$$

which gives

$$\begin{aligned} u(x,y) =& -c_2 y + F'(y/x), \\ v(x,y) =& c_2 x + (y/x)F'(y/x) - F(y/x), \end{aligned} \tag{63}$$

where F is an arbitrary function of one variable. A solution of the system (1) consists of the angle θ and the pressure σ defined by (39) with the velocities defined by (63). For example, if we choose the arbitrary function to be an elliptic function, that is

$$F(\xi) = \mathrm{cn}\left(\left(1 + \cosh(\arctan(b_2\xi))\right)^{-1}, \varrho\right), \qquad 0 < \rho^2 < 1,$$

and we set the parameters as $b_1 = 4\pi$, $c_2 = 0$, $\rho = 1/2$, then we can trace (see figure 4) an extrusion die for a feeding speed of $(U_0, V_0) = (0, -0.94)$ and an extraction speed $(U_1, v_1) = (0, -0.94)$. The curve C_1 on the figure 4 delimits the plasticity region at the mouth of the tool, while the x-axis does the same for the output of the tool. This type of tool could be used to undulate a plate. We can shape the tool by varying the parameters. For example, we can spread the bump by decreasing the parameter b_1. Moreover, one should note that if the modulus ϱ of the elliptic function is such that $0 \leq \varrho^2 \leq 1$, then the solution has one purely real and one purely imaginary period. For a real argument χ, we have the relations

$$-1 \leq \mathrm{cn}(\chi, \varrho) \leq 1.$$

(b). Suppose that $\zeta_2(\xi)$ is defined by (50) and for simplification we choose in particular

$$\zeta_3(\xi) = \frac{\zeta_1(\xi)}{\xi}. \tag{64}$$

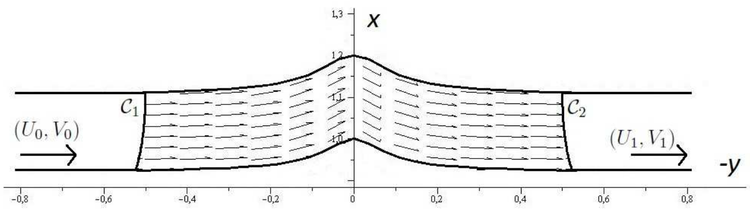

Figure 4. Extrusion die corresponding to the solution (39), (63).

Applying the mixed derivatives compatibility condition of f to the equations (43), (44), we get the following ODE for the function g:

$$g_{xx} - g_{yy} - 2\cot(2J(\xi))g_{xy} - \frac{4\xi J'(\xi)g_y}{x\sin^2(2J(\xi))} = 0. \tag{65}$$

By the change of variable

$$\xi(x, y) = y/x, \qquad \eta(x, y) = x^2 + y^2, \tag{66}$$

we reduce the PDE (65) in term of x and y, to the much simpler PDE in term of ξ and η,

$$g_{\xi\eta} + \frac{\xi g_\eta}{\xi^2 + 1} = 0, \tag{67}$$

which has the solution

$$g(\xi, \eta) = \zeta_4(\xi) + \frac{\zeta_5(\eta)}{\sqrt{\xi^2 + 1}}, \tag{68}$$

where ζ_4 and ζ_5 are arbitrary functions of one variable. Then, we find the solution for f by integration of the PDE (43), (44), with ζ_3 given by (64),

$$f(x, y) = -\int^{\xi(x,y)} \frac{\zeta_1(\xi) - \zeta_4'(\xi)}{\xi} d\xi - \frac{y\zeta_5(\eta(x, y))}{\sqrt{\eta(x, y)}} + c_2. \tag{69}$$

By the substitution of (68) and (69) in the equations (41), (42), we find that F is an arbitrary function of one variable and G is defined by

$$G(\xi(x, y)) = \int^{\xi(x,y)} (-\zeta_1(\xi) + F'(\xi)\xi)\, d\xi + c_3. \tag{70}$$

So, we introduce (68), (69) and (70) in (40) and after an appropriate redefinition of ζ_1, ζ_4 and ζ_5, the solution of (1.c), (1.d) is provided by

$$u(x, y) = K'(y/x) - yH(x^2+y^2) + c_2, \qquad v(x, y) = -K'(y/x) + \xi K(\xi) + xH(x^2+y^2) + c_1, \tag{71}$$

where H, K are arbitrary functions of one variable. The velocities (71), together with the angle and pressure defined by (39), solve the system (1). For example, a tool corresponding to the solution (71) with $H(\eta) = 2\exp(-0.1\eta)$, $K(\xi) = \xi$ and for feeding

and extraction speed given respectively by $(U_0, v_0) = (1.05, 0)$ and $(U_1, V_1) = (1.05, 0)$. It is shown in figure 5. The plasticity region limits correspond to the curves C_1 and C_2. This tool is symmetric under the reflection $x \mapsto -x$. Moreover, the top contour of the tool almost makes a complete loop, and this lets one suppose that we could make a ring in a material by extrusion.

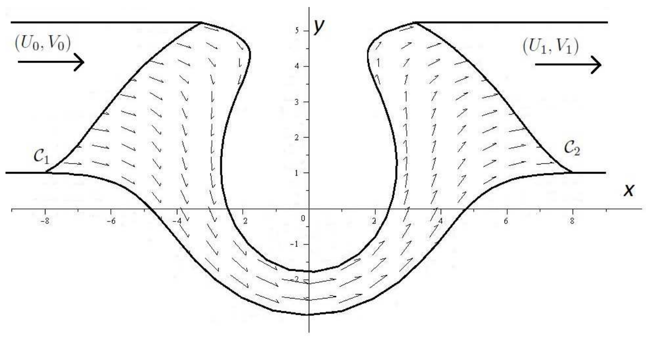

Figure 5. Extrusion die corresponding to the solution (39), (71).

3.3.3. Multiplicative separation for the velocities u and v when $c_1 \neq 0$. Consider the solutions of (1.a), (1.b), given by the angle $\theta(x, y) = J(y/x)$ with J defined by (36) and the pressure σ defined by (37). We require that the solutions for u and v be in the multiplicative separated form

$$u(x, y) = f(x, y)F(\xi(x, y)), \quad v(x, y) = g(x, y)G(\xi(x, y)), \tag{72}$$

where $\xi(x, y) = y/x$ and f, g, F, G are to be determined. The Ansatz (72) on the velocities brings the system (1.c), (1.d) to the form

$$\begin{aligned} &\left[f_y F + x^{-1} f F'(\xi) + g_x G(\xi) - x^{-1} \xi g G'(\xi)\right] \sin(2J(\xi)) \\ &+ \left[f_x F(\xi) - x^{-1} \xi f F' - g_y G - x^{-1} g G'\right] \cos(2J(\xi)) = 0, \\ &[f_x F(\xi) + g_y G(\xi)] - \xi f F'(\xi) + g G'(\xi) = 0. \end{aligned} \tag{73}$$

To reduce the PDE system (73) to an ODE system involving F and G in terms of ξ, we act with the operator $(1/2)(D_1 + D_2)$, defined by (4), and annihilate the function of ξ present in (73). This leads to conditions on f and g that do not involve F, G and their derivatives. There are three cases to consider, that is

$$\begin{aligned} &(a) \quad (1/2)(D_1 + D_2)\,(g_y/g) \neq 0, \qquad (D_1 + D_2)\left(\frac{(1/2)(D_1 + D_2)(f/g)}{(D_1 + D_2)(x g_y/g)}\right) \neq 0, \\ &(b) \quad (1/2)(D_1 + D_2)(x g_y/g) = 0, \\ &(c) \quad (1/2)(D_1 + D_2)\,(x g_y/g) \neq 0, \qquad (D_1 + D_2)\left(\frac{(1/2)(D_1 + D_2)(f/g)}{(D_1 + D_2)(x g_y/g)}\right) = 0. \end{aligned} \tag{74}$$

In this paper, we present the details for the cases (a) and (b).

(a). We first suppose that the condition (74.a) is satisfied. In this case, the function f must be a solution of the PDE system

$$\begin{aligned} f_x =& -x^{-1}\xi\zeta_1(\xi)f + \zeta_2(\xi)g_y - x^{-1}\left(\zeta_2(\xi)\zeta_1'(\xi) - \zeta_2'(\xi)\right)g, \\ f_y =& x^{-1}\zeta_1(\xi)f + \zeta_2(\xi)g_x - 2\zeta_2(\xi)\cot(2J(\xi))g_y \\ &+ \left(\zeta_2(\xi)\zeta_1'(\xi) - \zeta_2'(\xi)\right)\frac{\xi\sin(2J(\xi)) + 2\cos(2J(\xi))}{x\sin(2J(\xi))}g, \end{aligned} \tag{75}$$

where ζ_1, ζ_2 are two arbitrary functions of one variable. For the system (75) to be compatible, the function g must satisfy the PDE

$$\begin{aligned} & g_{xx} - 2\cot(2J(\xi))g_{xy} - g_{yy} + 2x^{-1}\zeta_2(\xi)^{-1}\left(\xi + \cot(2J(\xi))\right)\zeta_3(\xi)g_x \\ & + 2x^{-1}\left[\left(\xi - \cot(2J(\xi))\right)\zeta_3(\xi) - 2\xi\sin^{-2}(J(\xi))J'(\xi)\right]g_y \\ & + x^{-2}\zeta_2(\xi)^{-1}\Big([4\xi J'(\xi)\sin^{-2}(2J(\xi)) + \left(\xi^2 - 1 + 2\xi\cot(2J(\xi))\right)\zeta_1'(\xi) \\ & - 2\xi - 2\cot(2J(\xi))]\zeta_3(\xi) - \left(\xi^2 - 1 + 2\xi\cot(2J(\xi))\right)\zeta_3'(\xi)\Big)g = 0, \end{aligned} \tag{76}$$

where we used the notation $\zeta_3(\xi) = \zeta_2(\xi)\zeta_1'(\xi) - \zeta_2'(\xi)$ to shorten the expression. The equation (76) is difficult to solve for arbitrary ζ_1, ζ_2, but if we make the particular choice

$$\zeta_2(\xi) = \omega_2\exp(\zeta_1(\xi)), \qquad \omega_2 \in \mathbb{R}, \tag{77}$$

then the PDE (76) reduces to

$$g_{xx} - 2\cot(2J(\xi))g_{xy} - g_{yy} - 4x^{-1}\xi J'(\xi)\sin^{-2}(J(\xi))g_y = 0, \tag{78}$$

which is solved by the function

$$g(x,y) = \omega_3 x + \omega_4\cos(2J(y/x)) + \omega_5, \quad \omega_3, \omega_4, \omega_5 \in \mathbb{R}. \tag{79}$$

With g given by (79) and $\zeta_2(\xi)$ by (77), the system (75) is compatible. Consequently, f is expressed in term of a quadrature. We find

$$\begin{aligned} f(x,y) =& 2\exp(\zeta_1(y/x))\omega_2\omega_4\int^{y/x}\xi^{-1}\sin(2J(\xi))J'(\xi)d\xi \\ &+ \exp(\zeta_1(y/x))\left(\omega_2\omega_3 + 2\omega_2\omega_4 c_1\ln(y) + \omega_6\right) \end{aligned} \tag{80}$$

With f given by (80) and g by (79), the solutions for F and G of the system (73) are

$$F(\xi) = \omega_1\exp(-\zeta_1(\xi)), \qquad G(\xi) = \omega_2. \tag{81}$$

By introducing (79), (80), (81) in (72) and redefining the free parameters ω_i, $i = 1,2,3,4$, the solution of (1.c), (1.d) for the velocities u and v is

$$\begin{aligned} u(x,y) &= 2c_4\int^{y/x}\xi^{-1}\sin(2J(\xi))J'(\xi)d\xi + c_5 y + 2c_1c_4\ln(y) + c_6, \\ v(x,y) &= -c_5 x - c_4\cos(2J(y/x)) + c_7, \end{aligned} \tag{82}$$

where the c_i, $i = 1,\ldots,7$, are integration constants and J is defined by (36). So, we have a solution of the system (1) composed of the angle θ in the form (31) with J given implicitly by (36) together with the pressure σ (37) and the velocities (82).

(b). Suppose now that the condition(74.c) is satisfied. In this case, the solution g takes the form

$$g(x,y) = h_1(x)\zeta_1(\xi(x,y)) \tag{83}$$

and the function f must be a solution of

$$\begin{aligned} f_x =& x^{-1}\xi\zeta_2'(\xi)f + x^{-1}\zeta_3(\xi)h_1(x), \\ f_y =& -2x^{-1}h_1(x)\zeta_3(\xi)\cot(2J(\xi)) + e^{-\zeta_2(\xi)}\left(\int^{y/x} e^{\zeta_2(\xi)}\zeta_3(\xi)d\xi - \omega_2\right)h_1'(x) \\ & - x^{-1}\left[\zeta_2'(\xi) + \xi\zeta_3(\xi)\right]. \end{aligned} \tag{84}$$

We omit, due to its complexity, the expression of the compatibility condition on the mixed derivative of f relative to x and y. Nevertheless, making the specific choice

$$h_1(x) = 1, \qquad \zeta_2(\xi) = \ln\left(\frac{\sin(2J(\xi))}{(\xi^2-1)\sin(2J(\xi)) + 2\xi\cos(2J(\xi))}\right) - \ln(\zeta_3(\xi)), \tag{85}$$

the system (84) turns out to be compatible and the solution of (84) is

$$\begin{aligned} f(x,y) =& \zeta_3(y/x)\bigg(c_1^{-1}\left(1-(y/x)^2 - 2(y/x)\cot(2J(y/x))\right)\int^{y/x}\xi^{-1}\sin(2J(\xi))J'(\xi)d\xi \\ & + \left[1-(y/x)^2 - 2(y/x)\cot(2J(y/x))\right](\ln y - \omega_4)\bigg). \end{aligned} \tag{86}$$

The substitution of (86), (83) and (85) in the system (73) results in an ODE system for F and G, omitted due to its complexity, which has the solution

$$\begin{aligned} F(\xi) &= \frac{\omega_1\sin(2J(\xi))}{\zeta_3(\xi)\left((\xi^2-1)\sin(2J(\xi)) + 2\xi\cos(J(\xi))\right)}, \\ G(\xi) &= \zeta_1(\xi)^{-1}\left(\omega_1\cos(2J(\xi)) - \omega_2\right). \end{aligned} \tag{87}$$

We finally obtain a solution for the system (1.c), (1.d) by the substitution of (86) and (87), with $h_1(x)$, $\zeta_2(\xi)$, defined by (85), in (72). This leads to

$$\begin{aligned} u(x,y) &= -c_4\ln(y) - c_1^{-1}c_4\int^{y/x}\xi^{-1}\sin(2J(\xi))J'(\xi)d\xi + c_5, \\ v(x,y) &= (1/2)c_1^{-1}\omega_1\cos(2J(y/x)) + c_6, \qquad c_1, c_4, c_5, c_6 \in \mathbb{R}. \end{aligned} \tag{88}$$

So, the system (1) is solved by the angle θ in the form (31) with J implicitly defined by (36), together with the pressure σ (37) and the velocities (88).

3.3.4. Multiplicative separation for the velocities u and v when $c_1 = 0$. Consider now the case where $c_1 = 0$ in (35) so the solution of (1.a), (1.b), for θ and σ is (39). We suppose that the velocities u and v are in the form (72). Introducing this form for the velocities and θ defined by (39) in the equations (1.c), (1.d), leads to the system (73), which reduces to an ODE system for F and G if the functions f and g satisfy the condition (74). The three different constraints (74) must be considered separately.

(a). In the first case, where we consider that the conditions (74.a) are satisfied, the functions f and g must satisfy the system (75), (76). Changing the variables (x, y) to the new variables (ξ, η) defined by

$$\xi(x,y) = y/x, \qquad \eta(x,y) = x^2 + y^2, \tag{89}$$

and considering θ given by (39), the system (75),(76), becomes

$$\begin{aligned} &f_\xi = -\zeta_1(\xi) f - \xi^{-1}\zeta_2 g_\xi - \xi^{-1}\left(\zeta_2'(\xi) - \zeta_1(\xi)\zeta_2(\xi)\right) g, \\ &f_\eta = \xi\zeta_2(\xi) g_\eta, \end{aligned} \tag{90}$$

$$\frac{g_{\xi\eta}}{g_\eta} + \frac{\zeta_2'(\xi)}{\zeta_2(\xi)} + \zeta_1(\xi) + \xi(\xi^2+1)^{-1} = 0, \tag{91}$$

where $\zeta_1(\xi)$, $\zeta_2(\xi)$ are arbitrary functions of one variable. The solution of the system (90), (91), for f and g as functions of ξ and η is

$$\begin{aligned} f(\xi,\eta) =& \xi\zeta_2(\xi)\left(K(\xi) + \zeta_2(\xi)^{-1}(1+\xi^2)^{-1/2}\left(H(\eta)e^{-\int\zeta_1(\xi)d\xi}\right)\right) \\ & - e^{-\int\zeta_1(\xi)d\xi}\left(\int \xi^{-1}(\xi^2+1)\big[\zeta_2(\xi) + \left(\zeta_1(\xi) + \xi(\xi^2+1)^{-1}\zeta_2(\xi)\right)H(\xi)\right. \\ & \left. + \zeta_2(\xi)K'(\xi)\big]d\xi - \omega_1\right), \\ g(\xi,\eta) =& K(\xi) + \left(e^{\int\zeta_1(\xi)d\xi}\zeta_2(\xi)\sqrt{\xi^2+1}\right), \end{aligned} \tag{92}$$

where the functions $K(\xi)$ and $H(\eta)$ are arbitrary functions of one variable. Introducing f and g expressed in the initial variables x, y, by the substitution of (89) in (83), the system (73) is reduced to an ODE system for the functions F and G in term of ξ which has the solution

$$F(\xi) = \omega_1 e^{\int\zeta_1(\xi)d\xi}, \qquad G(\xi) = -\omega_1\zeta_2(\xi)e^{\int\zeta_1(\xi)d\xi}. \tag{93}$$

Finally, redefining

$$\zeta_2(\xi) = -(\omega_1 K(\xi))^{-1}\exp\left(\int\zeta_1(\xi)d\xi\right)Q(\xi), \qquad H(\eta) = \omega_1^{-1}\sqrt{\eta}P(\eta)$$

where $Q(\xi)$, $P(\eta)$ are arbitrary functions, and doing the substitution of (92) and (93) in (72), we obtain the solution of (1.c), (1.d), for velocities u and v given by

$$\begin{aligned} &u(x,y) = yP(x^2+y^2) - x^{-1}yQ(y/x)\int^{y/x}\xi^{-1}\left((\xi^2+1)Q'(\xi) + \xi Q(\xi)\right)d\xi + c_2, \\ &v(x,y) = Q(y/x) - xP(x^2+y^2), \end{aligned} \tag{94}$$

This solution for velocities together with θ and σ defined by (39) solves the initial system (1).

(b). Consider now that the functions f and g satisfy the constraint (74.b). Then they take the form

$$f(x,y) = \zeta_1(y/x), \qquad g(x,y) = \zeta_2(y/x), \tag{95}$$

where ζ_1 and ζ_2 are arbitrary functions of one variable. Since the functions f and g depend only on the symmetry variable and the velocities u and v have the form (72), it is equivalent to consider that

$$u(x,y) = F(\xi(x,y)), \qquad v(x,y) = G(\xi(x,y)). \tag{96}$$

If we suppose that u and v are in the form (96), then the solution of (1) consists of the angle θ and the pressure σ given by (39) together with the velocities

$$u(x,y) = F(y/x), \qquad v(x,y) = \int^{y/x} \xi F'(\xi) d\xi + c_3, \tag{97}$$

where F is an arbitrary function of one variable.

(c). The third case to consider is when f and g obey the conditions (74.c), so they take the form

$$f(x,y) = \omega_3 x^{1+\omega_2}(y/x)^{(1+\omega_2)/2}(x^{-1}(x^2+y^2))^{\omega_2/4}, \qquad g(x,y) = x^{1+\omega_2}\zeta_1(y/x), \tag{98}$$

where $\zeta_1(\xi)$ is an arbitrary function of one variable. Then we introduce (98) in (73) and solve for F and G. The solution is

$$F(\xi) = \omega_1 \zeta_1^{-1}(\xi)\xi^{(1-\omega_2)/2}(\xi^2+1)^{\omega_2/4}, \qquad G(\xi) = (\xi^2+1)^{\omega_2/2}. \tag{99}$$

Finally, substitution of f, g, F, G, given by (98) and (99), in (73) gives, after redefining the parameters ω_i in a convenient way, the solution for u and v of the equations (1.c), (1.d),

$$u(x,y) = c_3 y(x^2+y^2)^{\omega_2/2}, \qquad v(x,y) = -c_4 x(x^2+y^2)^{\omega_2/2}, \tag{100}$$

where c_3, c_4 are integration constants. The velocities u and v together with the angle θ and the pressure σ given by (39) constitute a solution for the system (1). This solution is just a subcase of the previous one corresponding to the condition (74.a) and the choice $Q(\xi) = 0$, $P(\eta) = \eta^{\omega_2/2}$ in (94).

4. Final remarks.

In this paper, we have obtained the infinitesimal generators which generate the Lie algebra of symmetries for the system (1) describing a planar flow of an ideal plastic material. The existence of generators P_1 to P_5, D_1, D_2, L, B_2, given in equation (4), together with generator X_1 of an infinite subalgebra, defined by (5), were already known in the literature [1]. However, we have shown that the symmetry group is completed by the addition of generators B_1, B_3, B_4, B_5, B_6, K, defined by (4) and by X_2

given to equation (5), which generate an infinite-dimensional subalgebra. We have seen that it is possible to include the generator K in the basis of a finite-dimensional Lie subalgebra only if no generator B_3 to B_6 and P_1 to P_4 appears. For this reason, we consider separately the case of the subalgebra $\mathcal{L}$ which excludes K and that of subalgebra $\mathcal{S}$ including K, defined by equations (8) and (9) respectively. For each of these two subalgebras, we have performed a classification of the subalgebras into conjugation classes under the action of the symmetry group using the methods described in [11, 12] (see Section 2 and the appendix). This classification is an important tool in the analysis of invariant and partially invariant solutions. A classification of the symmetry subalgebras of (1) has been performed in the past [1] for one-dimensional subalgebras. However, the classification presented here is more complete in the sense that it includes new infinitesimal generators and a classification of two-dimensional subalgebras, which can be used to obtain partially invariant solutions. In section 3, we have performed (as an example) symmetry reductions corresponding to one-dimensional subalgebras represented by newly found generators. For the first reduction, we have used the generator B_1 and for the second reduction the generator K, both defined in the list of generators (4). The symmetry reduction, using the invariants of B, leads to a new solution (see (21), (22)) defined in terms of an arbitrary function of $\xi = xu + yv$ and where the velocity fields are implicitly defined. For a particular choice of the arbitrary function, we have traced (in Figure 1) the shape of an extrusion die corresponding to this solution. The obtained solution has the particularity that the kinetic energy is constant along the flow. A similar analysis has been performed for the generator K in order to obtain a partially invariant solution. For this solution, the invariants given in equation (24) were used to add constraints which allow us to obtain a solution more easily. These considerations were illustrated by finding a particular solution of system (1) defined by equations (27), (28) et (30). An example of a velocity vector field and an example of an extrusion die have been traced respectively in figures 2 and 3. It should be noted that, to the vector field in Figure 2 (*i.e.* for the corresponding parameter values), we can associate a large family of extrusion dies, of which the one in Figure 3 is a particular choice. The contours have to be chosen in such a way that they correspond to flow lines to the extent that it is possible to trace two curves linking them which satisfy equation (2). These curves are the limit of the plasticity region.

An interesting observation concerning the generator K is that if we take the commutator of K with the generators $\{P_1, P_2, P_3, P_4\}$, we obtain the generators $\{B_3, B_4, B_5, B_6\}$. Repeating the procedure with generators $\{B_3, B_4, B_5, B_6\}$, we generate four new generators, and so on. An interesting fact is that, at each step, the new obtained generators can complete the base of the subalgebra $\mathcal{L}_i$ generated at the previous stage in order to form a new higher-dimensional (but still finite-dimensional) subalgebra (ensured by excluding K from the base). Consequently, it is always possible to enlarge a finite-dimensional subalgebra $\mathcal{L}_i$ that excludes K by increasing its base with the result of commutators $[K, Z]$ with $Z \in \mathcal{L}_i$. This gives us a chain of finite-subalgebras of the form

$\mathcal{L} \subset \mathcal{L}_1 \subset \mathcal{L}_2 \subset \ldots \subset \mathcal{L}_i \subset \ldots$. This subject will be addressed in future works.

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Appendix. List of two-dimensional representative subalgebras of $\mathcal{L}$

$\mathcal{L}_{2,1} = \{B_1, D_2\}$	$\mathcal{L}_{2,23} = \{B_1 + \epsilon D_1 + \delta P_5, L + aP_5\}$
$\mathcal{L}_{2,2} = \{B_1, D_2 + \lambda L + a_1 D_1 + a_2 P_5\}$	$\mathcal{L}_{2,24} = \{B_1 + \epsilon D_1 + aP_5, P_1\}$
$\mathcal{L}_{2,3} = \{B_1, D_2 + \lambda D_1\}$	$\mathcal{L}_{2,25} = \{B_1 + \epsilon B_5 + aB_6, B_3\}$
$\mathcal{L}_{2,4} = \{B_1, D_2 + 3D_1 + \delta P_5\}$	$\mathcal{L}_{2,26} = \{B_1 + \epsilon_1 B_5 + a_1 B_6, B_3 + a_2 P_1 + a_3 P_2 + \epsilon_2 P_4\}$
$\mathcal{L}_{2,5} = \{B_1, D_2 - D_1 + \delta P_5\}$	$\mathcal{L}_{2,27} = \{B_1 + \epsilon_1 B_5 + a_1 B_6 + a_2 P_4 + \epsilon_2 P_5,$ $B_3 + a_3 P_1 + a_4 P_2 + a_4 P_4\}$
$\mathcal{L}_{2,6} = \{B_1, D_2 + D_1 + \delta P_5\}$	$\mathcal{L}_{2,28} = \{B_1 + \epsilon B_5 + a_1 P_4, D_2 + 3D_1 + a_2 P_5\}$
$\mathcal{L}_{2,7} = \{B_1, D_2 + \delta P_5\}$	$\mathcal{L}_{2,29} = \{B_1 + \epsilon B_5 + a_1 P_4 + a_2 P_5, P_1\}$
$\mathcal{L}_{2,8} = \{B_1, L + a_1 D_1 + a_2 P_5\}$	$\mathcal{L}_{2,30} = \{B_1 + \epsilon P_3, P_5\}$
$\mathcal{L}_{2,9} = \{B_1, D_1 + aP_5\}$	$\mathcal{L}_{2,31} = \{B_1 + \epsilon P_3 + a_1 P_4, B_3 + a_2 P_2\}$
$\mathcal{L}_{2,10} = \{B_1, B_3\}$	$\mathcal{L}_{2,32} = \{B_1 + \epsilon_1 P_3 + a_1 P_4 + \epsilon_2 P_5, B_3 + a_2 P_2\}$
$\mathcal{L}_{2,11} = \{B_1, B_3 + aP_2 + \epsilon P_4\}$	$\mathcal{L}_{2,33} = \{B_1 + \epsilon_1 P_3 + a_1 P_4 + a_2 P_5, B_3 + a_3 P_2 + \epsilon_2 P_4\}$
$\mathcal{L}_{2,12} = \{B_1, B_3 + \epsilon_1 P_4 + \epsilon_2 P_5\}$	$\mathcal{L}_{2,34} = \{B_1 + \epsilon_1 P_3 + a_1 P_4 + a_2 P_5, B_3 + \epsilon_2 P_4 + \delta P_5\}$
$\mathcal{L}_{2,13} = \{B_1, B_3 + \epsilon P_5\}$	$\mathcal{L}_{2,35} = \{B_1 + \epsilon P_3 + a_1 P_4 + a_2 P_5, B_3 + \delta P_5\}$
$\mathcal{L}_{2,14} = \{B_1, P_1\}$	$\mathcal{L}_{2,36} = \{B_1 + \epsilon P_3 + aP_4 + \epsilon_2 P_5, P_1\}$
$\mathcal{L}_{2,15} = \{B_1, P_5\}$	$\mathcal{L}_{2,37} = \{B_1 + \epsilon_1 P_3 + a_1 P_4, P_1\}$
$\mathcal{L}_{2,16} = \{B_1 + \epsilon L, D_1 + a_1 L + a_2 P_5\}$	$\mathcal{L}_{2,38} = \{B_1 + \epsilon_1 P_4, P_1\}$
$\mathcal{L}_{2,17} = \{B_1 + \epsilon L + aD_1, P_5\}$	$\mathcal{L}_{2,39} = \{B_1 + \epsilon_1 P_4 + \epsilon_2 P_5, P_1\}$
$\mathcal{L}_{2,18} = \{B_1 + \epsilon L + \delta P_5, D_1 + a_1 L + a_2 P_5\}$	$\mathcal{L}_{2,40} = \{B_1 + \epsilon P_5, L + a_1 D_1 + a_2 P_5\}$
$\mathcal{L}_{2,19} = \{B_1 + \epsilon D_1, L + aP_5\}$	$\mathcal{L}_{2,41} = \{B_1 + \epsilon P_5, B_3 + a_1 P_2 + a_2 P_4\}$
$\mathcal{L}_{2,20} = \{B_1 + \epsilon D_1 + a_1 P_5, B_3 + a_2 P_2\}$	$\mathcal{L}_{2,42} = \{B_1 + \epsilon P_5, B_3 + aP_4 + \epsilon_2 P_5\}$
$\mathcal{L}_{2,21} = \{B_1 + \epsilon D_1 + a_1 P_5, B_3 + a_2 P_2 + \epsilon P_4\}$	$\mathcal{L}_{2,43} = \{B_1 + \epsilon P_5, P_1\}$
$\mathcal{L}_{2,22} = \{B_1 + \epsilon D_1, P_5\}$	

Table 6. List of 2-dimensional representative subalgebras of $\mathcal{L}$ that have a nonzero component B_1 but with no B_2 component. The parameters are $\epsilon = \pm 1$ and $a, \lambda, \delta, \in \mathbb{R}$, where $\delta \neq 0$, $\lambda > 0$.

$\mathcal{L}_{2,44} = \{D_2, L + a_1D_1 + a_2P_5\}$	$\mathcal{L}_{2,75} = \{D_2 + D_1 + \epsilon B_5 + \delta P_5, P_1\}$
$\mathcal{L}_{2,45} = \{D_2, D_1 + aP_5\}$	$\mathcal{L}_{2,76} = \{D_2 + D_1 + \epsilon B_5 + \delta P_5, P_u\}$
$\mathcal{L}_{2,46} = \{D_2, B_3 + aP_2\}$	$\mathcal{L}_{2,77} = \{D_2 + D_1 + \epsilon B_6, B_5\}$
$\mathcal{L}_{2,47} = \{D_2, B_5 + aP_4\}$	$\mathcal{L}_{2,78} = \{D_2 + D_1 + \epsilon B_6, B_5 + \delta P_u + aP_v\}$
$\mathcal{L}_{2,48} = \{D_2, P_1\}$	$\mathcal{L}_{2,79} = \{D_2 + D_1 + \epsilon B_6, B_5 + \delta P_v\}$
$\mathcal{L}_{2,49} = \{D_2, P_3\}$	$\mathcal{L}_{2,80} = \{D_2 + D_1 + \epsilon B_6 + \delta P_3, B_5 + a_1P_3 + a_2P_4\}$
$\mathcal{L}_{2,50} = \{D_2, P_5\}$	$\mathcal{L}_{2,81} = \{D_2 + D_1 + \epsilon P_3 + aP_4, B_3\}$
$\mathcal{L}_{2,51} = \{D_2 + \lambda L, D_1 + aL\}$	$\mathcal{L}_{2,82} = \{D_2 + D_1 + \epsilon P_3 + aP_4, B_3 + \delta P_2\}$
$\mathcal{L}_{2,52} = \{D_2 + \lambda L + aD_1, P_5\}$	$\mathcal{L}_{2,83} = \{D_2 + D_1 + \epsilon P_3 + aP_4, B_5\}$
$\mathcal{L}_{2,53} = \{D_2 - D_1, P_1\}$	$\mathcal{L}_{2,84} = \{D_2 + D_1 + \epsilon P_3 + aP_4, B_5 + \delta P_4\}$
$\mathcal{L}_{2,54} = \{D_2 - D_1, B_3 + \epsilon P_5\}$	$\mathcal{L}_{2,85} = \{D_2 + D_1 + \epsilon P_3 + aP_4, B_5 + \delta P_5\}$
$\mathcal{L}_{2,55} = \{D_2 - D_1 + \epsilon B_3, P_1\}$	$\mathcal{L}_{2,86} = \{D_2 + D_1 + \epsilon P_3 + aP_4, P_1\}$
$\mathcal{L}_{2,56} = \{D_2 - D_1 + \epsilon B_3, P_3\}$	$\mathcal{L}_{2,87} = \{D_2 + D_1 + \epsilon P_3 + aP_4, P_3\}$
$\mathcal{L}_{2,57} = \{D_2 - D_1 + \epsilon B_3 + a_1P_2 + a_2P_5, P_1\}$	$\mathcal{L}_{2,88} = \{D_2 + D_1 + \epsilon P_4, B_3 + aP_2\}$
$\mathcal{L}_{2,58} = \{D_2 - D_1 + \epsilon P_1 + a_1P_2, B_3 + a_2P_2\}$	$\mathcal{L}_{2,89} = \{D_2 + D_1 + \epsilon P_4, B_5 + aP_v\}$
$\mathcal{L}_{2,59} = \{D_2 - D_1 + \epsilon P_1 + aP_2, B_3 + \delta P_5\}$	$\mathcal{L}_{2,90} = \{D_2 + D_1 + \epsilon P_4, B_5 + \delta P_5\}$
$\mathcal{L}_{2,60} = \{D_2 - D_1 + \epsilon P_1 + aP_2, B_5\}$	$\mathcal{L}_{2,91} = \{D_2 + D_1 + \epsilon P_4, P_1\}$
$\mathcal{L}_{2,61} = \{D_2 - D_1 + \epsilon P_1 + aP_2, B_5 + \delta P_4\}$	$\mathcal{L}_{2,92} = \{D_2 + D_1 + \epsilon P_4, P_3\}$
$\mathcal{L}_{2,62} = \{D_2 - D_1 + \epsilon P_1 + aP_2, P_3\}$	$\mathcal{L}_{2,93} = \{D_2 + D_1 + \delta P_5, P_3\}$
$\mathcal{L}_{2,63} = \{D_2 - D_1 + \epsilon P_1 + aP_2, P_5\}$	$\mathcal{L}_{2,94} = \{D_2 + \lambda D_1, L + aP_5\}$
$\mathcal{L}_{2,64} = \{D_2 - D_1 + \epsilon P_2, B_3 + a_2P_2\}$	$\mathcal{L}_{2,95} = \{D_2 + \lambda D_1, B_3 + aP_4\}$
$\mathcal{L}_{2,65} = \{D_2 - D_1 + \epsilon P_2, B_3 + \delta P_5\}$	$\mathcal{L}_{2,96} = \{D_2 + \lambda D_1, B_5 + aP4\}$
$\mathcal{L}_{2,66} = \{D_2 - D_1 + \epsilon P_2, B_5 + aP_4\}$	$\mathcal{L}_{2,97} = \{D_2 + \lambda D_1, P_3\}$
$\mathcal{L}_{2,67} = \{D_2 - D_1 + \epsilon P_2, P_3\}$	$\mathcal{L}_{2,98} = \{D_2 + \lambda D_1, P_5\}$
$\mathcal{L}_{2,68} = \{D_2 - D_1 + \epsilon P_2, P_5\}$	$\mathcal{L}_{2,99} = \{D_2 + \lambda D_1 + \delta P_5, L + aP_5\}$
$\mathcal{L}_{2,69} = \{D_2 - D_1 + \delta P_5, P_3\}$	$\mathcal{L}_{2,100} = \{D_2 + \lambda D_1 + aP_5, P_1\}$
$\mathcal{L}_{2,70} = \{D_2 + D_1, P_2\}$	$\mathcal{L}_{2,101} = \{D_2 + \lambda P_5, L + a_1D_1 + a_2P_5\}$
$\mathcal{L}_{2,71} = \{D_2 + D_1, B_5 + \epsilon P_5\}$	$\mathcal{L}_{2,102} = \{D_2 + \lambda P_5, D_1 + aP_5\}$
$\mathcal{L}_{2,72} = \{D_2 + D_1 + \epsilon B_5, P_1\}$	$\mathcal{L}_{2,103} = \{D_2 + \lambda P_5, P_1\}$
$\mathcal{L}_{2,73} = \{D_2 + D_1 + \epsilon B_5 + \delta P_4, P_1\}$	$\mathcal{L}_{2,104} = \{D_2 + \lambda P_5, P_3\}$
$\mathcal{L}_{2,74} = \{D_2 + D_1 + \epsilon B_5 + aP_4, P_u\}$	

Table 7. List of 2-dimensional representative subalgebras of $\mathcal{L}$ that have a nonzero component D_2 but with no component in $\{B_1, B_2\}$. The parameters are $\epsilon = \pm 1$ and $a, \lambda, \delta, \in \mathbb{R}$, where $\delta \neq 0$, $\lambda > 0$.

$\mathcal{L}_{2,105} = \{B_1 - B_2, P_5\}$	$\mathcal{L}_{2,114} = \{B_1 - B_2 + \epsilon_1 L + \delta P_3, P_1 - \epsilon_1 P_3 + \epsilon_2 P_5\}$
$\mathcal{L}_{2,106} = \{B_1 - B_2 + \delta L, D_1 + aL\}$	$\mathcal{L}_{2,115} = \{B_1 - B_2 + \epsilon_1 L + aP_3 + \delta P_4,$ $P_1 - \epsilon_1 P_3 + \epsilon_2 P_5\}$
$\mathcal{L}_{2,107} = \{B_1 - B_2 + \epsilon_1 L, P_1 - \epsilon_1 P_3 + \epsilon_2 P_5\}$	$\mathcal{L}_{2,116} = \{B_1 - B_2 + \epsilon_1 L + a_1 P_3 + a_2 P_4 + \delta P_5,$ $P_1 - \epsilon_1 P_3 + \epsilon P_5\}$
$\mathcal{L}_{2,108} = \{B_1 - B_2 + \epsilon L + aD_1, P_1 - \epsilon P_3\}$	$\mathcal{L}_{2,117} = \{B_1 - B_2 + \delta D_1, P_5\}$
$\mathcal{L}_{2,109} = \{B_1 - B_2 + \delta L + aD_1, P_5\}$	$\mathcal{L}_{2,118} = \{B_1 - B_2 + \delta D_1 + a_1 P_5, L + a_2 P_5\}$
$\mathcal{L}_{2,110} = \{B_1 - B_2 + \epsilon L + a_1 D_1 + a_2 P_1 + a_2 P_2,$ $B_3 - \epsilon B_5 + a_4 P_2 + a_5 P_4 + a_6 P_5\}$	$\mathcal{L}_{2,119} = \{B_1 - B_2 + a_1 P_5, L + a_2 D_1 + a_3 P_5\}$
$\mathcal{L}_{2,111} = \{B_1 - B_2 + \epsilon L + aD_1 + a_1 P_1 + a_2 P_2,$ $B_4 - \epsilon B_6 + a_3 P_1 + a_4 P_3\}$	$\mathcal{L}_{2,120} = \{B_1 - B_2 + a_1 P_5, D_1 + a_2 P_5\}$
$\mathcal{L}_{2,112} = \{B_1 - B_2 + \epsilon L + aD_1 + \delta P_5, P_1 - \epsilon P_3\}$	$\mathcal{L}_{2,121} = \{L + aD_1, P_5\}$
$\mathcal{L}_{2,113} = \{B_1 - B_2 + \epsilon_1 L + \epsilon_2 P_1, P_5\}$	$\mathcal{L}_{2,122} = \{L + a_1 P_5, D_1 + a_2 P_5\}$

Table 8. List of 2-dimensional representative subalgebras of $\mathcal{L}$ that have a nonzero component B_2. The parameters are $\epsilon = \pm 1$ and $a, \lambda, \delta, \in \mathbb{R}$, where $\delta \neq 0$, $\lambda > 0$.

$\mathcal{L}_{2,123} = \{D_1, B_3\}$	$\mathcal{L}_{2,127} = \{D_1, B_3 + aP_2 + \epsilon P_4\}$
$\mathcal{L}_{2,124} = \{D_1, B_3 + \epsilon B_5 + a_1 P_1 + a_2 P_2 + a_3 P_4\}$	$\mathcal{L}_{2,128} = \{D_1 + aP_5, P_1 + \epsilon P_3\}$
$\mathcal{L}_{2,125} = \{D_1, B_3 + \epsilon P_3\}$	$\mathcal{L}_{2,129} = \{D_1 + aP_5, P_3\}$
$\mathcal{L}_{2,126} = \{D_1, B_3 + a_1 P_2 + \epsilon P_3 + a_2 P_4\}$	

Table 9. List of 2-dimensional representative subalgebras of $\mathcal{L}$ that do not have components in $\{B_1, D_2, B_2, L\}$ with a nonzero D_1 component. The parameters are $\epsilon = \pm 1$ and $a, \lambda, \delta, \in \mathbb{R}$, where $\delta \neq 0$, $\lambda > 0$.

$\mathcal{L}_{2,130} = \{B_3, P_1 + aP_2\}$	$\mathcal{L}_{2,163} = \{B_3 + \epsilon B_5 + \delta P_5 + aP_4, P_2\}$
$\mathcal{L}_{2,131} = \{B_3 + \epsilon_1 B_5, P_1 + aP_2 + \epsilon_2 P_3\}$	$\mathcal{L}_{2,164} = \{B_3 + \epsilon B_5 + aP_4, P_2\}$
$\mathcal{L}_{2,132} = \{B_3 + \epsilon_1 B_5, P_1 + \epsilon_2 P_3\}$	$\mathcal{L}_{2,165} = \{B_3 + \epsilon B_5 + aP_4, P_2\}$
$\mathcal{L}_{2,133} = \{B_3 + \epsilon_1 B_5, P_2 + \epsilon_2 P_3\}$	$\mathcal{L}_{2,166} = \{B_3 + \epsilon B_5 + \delta P_3 + aP_4, P_2 + \epsilon_2 P_3\}$
$\mathcal{L}_{2,134} = \{B_3 + \epsilon P_3 + aP_4, P_1 + \delta P_2\}$	$\mathcal{L}_{2,167} = \{B_3 + \epsilon_1 B_5 + \delta P_4, P_2 + \epsilon_2 P_3\}$
$\mathcal{L}_{2,135} = \{B_3 + \epsilon P_4, P_1 + \delta P_2\}$	$\mathcal{L}_{2,168} = \{B_3 + \epsilon_1 B_5 + \delta P_5, P_2 + \epsilon_2 P_3\}$
$\mathcal{L}_{2,136} = \{B_3 + a_1 P_2, P_1\}$	$\mathcal{L}_{2,169} = \{B_3 + \epsilon B_5 + \delta P_2 + aP_4, P_3\}$
$\mathcal{L}_{2,137} = \{B_3 + a_1 P_2 + \epsilon P_3 + a_2 P_4, P_1\}$	$\mathcal{L}_{2,170} = \{B_3 + \epsilon B_5 + aP_4, P_3\}$
$\mathcal{L}_{2,138} = \{B_3 + aP_2 + \epsilon P_4, P_1\}$	$\mathcal{L}_{2,171} = \{B_3 + \epsilon B_5 + \delta P_5, P_3\}$
$\mathcal{L}_{2,139} = \{B_3 + \epsilon P_5, P_1 + aP_2\}$	$\mathcal{L}_{2,172} = \{B_3 + a_1 P_1 + a_2 P_2, B_3 + a_3 P_1 + \epsilon P_3 + a_4 P_4\}$
$\mathcal{L}_{2,140} = \{B_3 + \delta P_2 + a_1 P_4, P_1 + \epsilon P_3 + a_2 P_4\}$	$\mathcal{L}_{2,173} = \{B_3 + a_1 P_1 + a_2 P_2, B_4 + a_3 P_1 + \epsilon P_4\}$
$\mathcal{L}_{2,141} = \{B_3 + a_1 P_4, P_1 + \epsilon P_3 + a_2 P_4\}$	$\mathcal{L}_{2,174} = \{B_3 + a_1 P_1 + a_2 P_2, B_4 + a_3 P_1\}$
$\mathcal{L}_{2,142} = \{B_3 + \epsilon P_5, P_1 + \epsilon P_3 + aP_4\}$	$\mathcal{L}_{2,175} = \{B_3 + a_1 P_1 + a_2 P_2 + \epsilon P_3 + a_3 P_4, B_4 + a_4 P_1 + a_5 P_3 + a_6 P_4\}$
$\mathcal{L}_{2,143} = \{B_3 + \delta P_3 + aP_4, P_2 + \epsilon P_4\}$	$\mathcal{L}_{2,176} = \{B_3 + a_1 P_1 + a_2 P_2 + \epsilon P_4, B_4 + a_3 P_1 + a_4 P_3 + a_5 P_5\}$
$\mathcal{L}_{2,144} = \{B_3 + \delta P_4, P_2 + \epsilon P_4\}$	$\mathcal{L}_{2,177} = \{B_3 + a_1 P_1 + a_2 P_2, B_4 + \epsilon B_6 + a_3 P_1 + a_4 P_2 + a_5 P_3\}$
$\mathcal{L}_{2,145} = \{B_3 + aP_5, P_2 + \epsilon P_4\}$	$\mathcal{L}_{2,178} = \{B_3 + a_1 P_1 + a_2 P_2, B_5 + a_3 B_6 + a_4 P_4\}$
$\mathcal{L}_{2,146} = \{B_3 + a_1 P_2, P_3 + a_2 P_4\}$	$\mathcal{L}_{2,179} = \{B_3 + a_1 P_1 + a_2 P_2, B_5 + a_3 B_6 + \epsilon P_x + a_4 P_2 + a_5 P_5\}$
$\mathcal{L}_{2,147} = \{B_3 + a_1 P_2 + \epsilon P_4, P_3 + a_2 P_4\}$	$\mathcal{L}_{2,180} = \{B_3 + a_1 P_x + a_2 P_2, B_5 + a_3 B_6 + \epsilon P_2 + a_4 P_4\}$
$\mathcal{L}_{2,148} = \{B_3 + \epsilon P_5, P_3 + aP_4\}$	$\mathcal{L}_{2,181} = \{B_3 + a_1 P_1 + a_2 P_2 + \epsilon P_3 + a_3 P_v, B_5 + a_4 B_6 + a_5 P_1 + a_6 P_2 + a_7 P_4\}$
$\mathcal{L}_{2,149} = \{B_3 + aP_2, P_4\}$	$\mathcal{L}_{2,182} = \{B_3 + a_1 P_1 + a_2 P_2 + \epsilon P_4, B_5 + a_3 B_6 + a_4 P_1 + a_5 P_2 + a_6 P_4\}$
$\mathcal{L}_{2,150} = \{B_3 + aP_2 + \epsilon P_3, P_4\}$	$\mathcal{L}_{2,183} = \{B_3 + a_1 P_1 + a_2 P_2 + \epsilon P_3 + a_3 P_4, B_6 + a_4 P_1 + a_5 P_2 + a_6 P_3\}$
$\mathcal{L}_{2,151} = \{B_3 + \epsilon P_5, P_4\}$	$\mathcal{L}_{2,184} = \{B_3 + a_1 P_1 + a_2 P_2 + \epsilon P_4, B_6 + a_3 P_1 + a_4 P_2 + a_5 P_3\}$
$\mathcal{L}_{2,152} = \{B_3 + \epsilon_1 B_5 + \delta P_2, P_1 + aP_2\}$	$\mathcal{L}_{2,185} = \{B_3 + a_1 P_1 + a_2 P_2, B_6 + \epsilon P_1 + a_3 P_2 + a_4 P_3\}$
$\mathcal{L}_{2,153} = \{B_3 + \epsilon_1 B_5 + a_1 P_2 + \epsilon_2 P_4, P_1 + a_2 P_y\}$	$\mathcal{L}_{2,186} = \{B_3 + a_1 P_1 + a_2 P_2, B_6 + \epsilon P_2 + a_4 P_3\}$
$\mathcal{L}_{2,154} = \{B_3 + \epsilon B_5 + a_1 P_5, P_1 + a_2 P_2\}$	$\mathcal{L}_{2,187} = \{B_3 + a_1 P_1 + a_2 P_2, B_6 + a_3 P_3\}$
$\mathcal{L}_{2,155} = \{B_3 + \epsilon_1 B_5 + \delta P_2 + a_1 P_3 + a_2 P_4, P_1 + a_3 P_2 + \epsilon_2 P_3\}$	$\mathcal{L}_{2,188} = \{B_3 + a_1 B_4 + a_2 P_1 + a_2 P_2 + \epsilon P_3 + a_3 P_4, B_5 + a_4 P_1 + a_5 P_2 + a_6 P_4\}$
$\mathcal{L}_{2,156} = \{B_3 + \epsilon_1 B_5 + \delta P_3 + a_1 P_4, P_1 + a_2 P_y + \epsilon_2 P_3\}$	$\mathcal{L}_{2,189} = \{B_3 + a_1 B_4 + a_2 P_1 + a_3 P_2 + \epsilon P_4, B_5 + a_4 P_1 + a_4 P_2 + a_5 P_4\}$
$\mathcal{L}_{2,157} = \{B_3 + \epsilon_1 B_5 + \delta P_4, P_1 + a_2 P_2 + \epsilon_2 P_u\}$	$\mathcal{L}_{2,190} = \{B_3 + a_1 B_4 + a_2 P_1 + a_2 P_2, B_5 + \epsilon P_1 + aP_2 + aP_4\}$
$\mathcal{L}_{2,158} = \{B_3 + \epsilon_1 B_5 + \delta P_5, P_1 + aP_2 + \epsilon_2 P_3\}$	$\mathcal{L}_{2,191} = \{B_3 + a_1 B_4 + a_2 P_1 + a_2 P_2, B_5 + \epsilon P_4 + a_3 P_4\}$
$\mathcal{L}_{2,159} = \{B_3 + \epsilon_1 B_5 + \delta P_2 + a_1 P_3 + a_2 P_4, P_1 + \epsilon_2 P_3\}$	$\mathcal{L}_{2,192} = \{B_3 + a_1 B_4 + a_2 P_1 + a_3 P_2, B_5 + a_4 P_4\}$
$\mathcal{L}_{2,160} = \{B_3 + \epsilon_1 B_5 + \delta P_3 + aP_v, P_1 + \epsilon_2 P_u\}$	$\mathcal{L}_{2,193} = \{B_3 + \epsilon B_5 + a_1 P_1 + a_2 P_2 + a_3 P_3 + a_4 P_4,$ $B_4 + a_5 B_5 + a_6 B_6 + a_7 P_1 + a_8 P_3 + a_9 P_4\}$
$\mathcal{L}_{2,161} = \{B_3 + \epsilon_1 B_5 + \delta P_4, P_1 + \epsilon_2 P_3\}$	$\mathcal{L}_{2,194} = \{B_3 + \epsilon B_5 + a_1 P_1 + a_2 P_2 + a_3 P_3 + a_4 P_4, B_5 + a_5 B_6 + a_6 P_1 + a_7 P_2 + a_8 P_4\}$
$\mathcal{L}_{2,162} = \{B_3 + \epsilon_1 B_5 + \delta P_5, P_1 + \epsilon_2 P_u\}$	$\mathcal{L}_{2,195} = \{B_3 + \epsilon B_5 + a_1 P_1 + a_2 P_2 + a_3 P_3 + a_4 P_4, B_6 + a_5 P_1 + a_6 P_2 + a_7 P_3\}$

Table 10. List of 2-dimensional representative subalgebras of $\mathcal{L}$ that do not have components in $\{B_1, D_2, B_2, L, D_1\}$ with a nonzero B_3 component. The parameters are $\epsilon = \pm 1$ and $a, \lambda, \delta, \in \mathbb{R}$, where $\delta \neq 0$, $\lambda > 0$.

$\mathcal{L}_{2,196} = \{B_5, B_6\}$	$\mathcal{L}_{2,209} = \{B_5 + aP_4, P_2 + \epsilon P_4\}$
$\mathcal{L}_{2,197} = \{B_5, P_3 + aP_4\}$	$\mathcal{L}_{2,210} = \{B_5 + \delta P_5, P_2 + \epsilon P_4\}$
$\mathcal{L}_{2,198} = \{B_5, P_4\}$	$\mathcal{L}_{2,211} = \{B_5 + \epsilon P_1 + a_1 P_2 + a_2 P_4, P_3 + a_3 P_v\}$
$\mathcal{L}_{2,199} = \{B_5 + \epsilon P_2 + a_1 P_4, P_1 + a_2 P_2\}$	$\mathcal{L}_{2,212} = \{B_5 + \epsilon P_2 + a_1 P_4, P_3 + a_2 P_4\}$
$\mathcal{L}_{2,200} = \{B_5 + a_1 P_4, P_1 + a_2 P_2\}$	$\mathcal{L}_{2,213} = \{B_5 + \delta P_4, P_3 + a_2 P_4\}$
$\mathcal{L}_{2,201} = \{B_5 + \epsilon P_5, P_1 + aP_2\}$	$\mathcal{L}_{2,214} = \{B_5 + \epsilon P_5, P_3 + aP_4\}$
$\mathcal{L}_{2,202} = \{B_5 + \epsilon_1 P_2 + a_1 P_4, P_1 + a_2 P_2 + \epsilon_2 P_3\}$	$\mathcal{L}_{2,215} = \{B_5 + \epsilon P_1 + aP_2, P_4\}$
$\mathcal{L}_{2,203} = \{B_5 + a_1 P_4, P_1 + a_2 P_2 + \epsilon P_3\}$	$\mathcal{L}_{2,216} = \{B_5 + \epsilon P_2, P_4\}$
$\mathcal{L}_{2,204} = \{B_5 + \epsilon_1 P_5, P_1 + aP_2 + \epsilon_2 P_3\}$	$\mathcal{L}_{2,217} = \{B_5 + \epsilon P_5, P_4\}$
$\mathcal{L}_{2,205} = \{B_5 + \epsilon P_1 + aP_4, P_2\}$	$\mathcal{L}_{2,218} = \{B_5 + a_1 P_3 + a_2 P_4, B_6 + \epsilon P_1 + a_3 P_1 + a_3 P_2 + a_4 P_3\}$
$\mathcal{L}_{2,206} = \{B_5 + aP_4, P_2\}$	$\mathcal{L}_{2,219} = \{B_5 + a_1 P_3 + a_2 P_4, B_6 + \epsilon P_2 + a_3 P_3\}$
$\mathcal{L}_{2,207} = \{B_5 + \epsilon P_5, P_2\}$	$\mathcal{L}_{2,220} = \{B_5 + \epsilon P_1 + a_1 P_2 + a_2 P_3 + a_3 P_4, B_6 + a_4 P_1 + a_5 P_2 + a_6 P_3\}$
$\mathcal{L}_{2,208} = \{B_5 + \delta P_1 + aP_4, P_2 + \epsilon P_4\}$	$\mathcal{L}_{2,221} = \{B_5 + \epsilon P_2 + a_1 P_3 + a_2 P_4, B_6 + a_3 P_1 + a_4 P_2\}$

Table 11. List of 2-dimensional representative subalgebras of $\mathcal{L}$ that do not have components in $\{B_1, D_2, B_2, L, D_1, B_3, B_4\}$ with a nonzero B_5 component. The parameters are $\epsilon = \pm 1$ and $a, \lambda, \delta, \in \mathbb{R}$, where $\delta \neq 0$, $\lambda > 0$.

$\mathcal{L}_{2,222} = \{P_1, P_5\}$	$\mathcal{L}_{2,232} = \{P_1 + \epsilon P_5, P_4\}$
$\mathcal{L}_{2,223} = \{P_1 + aP_2, P_5\}$	$\mathcal{L}_{2,233} = \{P_2, P_5\}$
$\mathcal{L}_{2,224} = \{P_1 + aP_2 + \epsilon_1 P_5, P_2 + \epsilon_2 P_5\}$	$\mathcal{L}_{2,234} = \{P_2 + a_1 P_3 + a_2 P_4, P_5\}$
$\mathcal{L}_{2,225} = \{P_1 + \epsilon P_3, P_5\}$	$\mathcal{L}_{2,235} = \{P_2 + \epsilon P_5, P_3 + \epsilon_2 P_5\}$
$\mathcal{L}_{2,226} = \{P_1 + \epsilon_1 P_3 + \epsilon_2 P_5, P_2 + a_1 P_3 + a_2 P_4 + \delta P_5\}$	$\mathcal{L}_{2,236} = \{P_2 + \epsilon_1 P_5, P_3 + \epsilon_2 P_5\}$
$\mathcal{L}_{2,227} = \{P_1 + \epsilon_1 P_3 + \epsilon_2 P_5, P_v + \delta P_5\}$	$\mathcal{L}_{2,237} = \{P_3, P_5\}$
$\mathcal{L}_{2,228} = \{P_1 + \epsilon_1 P_5, P_2 + \epsilon_2 P_4\}$	$\mathcal{L}_{2,238} = \{P_3 + aP_4, P_3\}$
$\mathcal{L}_{2,229} = \{P_1 + \epsilon P_5, P_2 + \delta P_5\}$	$\mathcal{L}_{2,239} = \{P_3 + \epsilon P_5, P_4 + \delta P_5\}$
$\mathcal{L}_{2,230} = \{P_1 + \epsilon P_5, P_3 + \delta P_4\}$	$\mathcal{L}_{2,240} = \{P_4, P_5\}$
$\mathcal{L}_{2,231} = \{P_1 + \epsilon_1 P_5, P_3 + \epsilon_2 P_5\}$	

Table 12. List of 2-dimensional representative subalgebras of $\mathcal{L}$ corresponding to translations with the P_5 component. The parameters are $\epsilon = \pm 1$ and $a, \lambda, \delta, \in \mathbb{R}$, where $\delta \neq 0$, $\lambda > 0$.

$\mathcal{L}_{2,241} = \{P_1, P_2\}$	$\mathcal{L}_{2,246} = \{P_1 + \epsilon P_3, P_2 + a_1 P_3 + a_2 P_4\}$
$\mathcal{L}_{2,242} = \{P_1, P_2 + \epsilon P_4\}$	$\mathcal{L}_{2,247} = \{P_1 + \epsilon P_3, P_3 + aP_4\}$
$\mathcal{L}_{2,243} = \{P_1, P_3 + aP_4\}$	$\mathcal{L}_{2,248} = \{P_1 + \epsilon P_3, P_4\}$
$\mathcal{L}_{2,244} = \{P_1, P_4\}$	$\mathcal{L}_{2,249} = \{P_2, P_3\}$
$\mathcal{L}_{2,245} = \{P_1 + aP_2, P_3\}$	$\mathcal{L}_{2,250} = \{P_3, P_4\}$

Table 13. List of 2-dimensional representative subalgebras of $\mathcal{L}$ corresponding to translations without the P_5 component. The parameters are $\epsilon = \pm 1$ and $a, \lambda, \delta, \in \mathbb{R}$, where $\delta \neq 0$, $\lambda > 0$.